1 Quasi-isometry rigidity for hyperbolic buildings

After M. Bourdon and H. Pajot
A summary written by Seonhee Lim

Abstract

We summarize Boundon-Pajot's quasi-isometry rigidity for some negatively-curved buildings.

1.1 Introduction

As the first application of quasiconformal analysis on boundaries of hyperbolic groups in this summer school, let us start with a general question taken from the survey paper of Kleiner [6].

Question. Given a finitely generated group G, what is its quasi-isometry group $QI(G)$?

Recall that the quasi-isometry group $QI(X)$ of a metric space X is the set of equivalence classes of quasi-isometries $f : X \to X$, where two quasi-isometries f_1, f_2 are equivalent iff $\sup_x d(f_1(x), f_2(x)) < \infty$ (here we consider G as a metric space with a word metric). One approach to this question, which has been the most successful one, is to find an 'optimal' space X quasi-isometric to G and show that the natural map $\mathrm{Isom}(X) \to QI(X)$ is an isomorphism. This statement was proved in various settings, started by Pansu (for quaternionic and Cayley hyperbolic spaces), followed by results of Kleiner and Leeb (for products of irreducible affine buildings and symmetric spaces of dimension ≥ 2), Kapovich and Schwartz (for universal covers of compact locally symmetric spaces of dimension ≥ 3).

Bourdon and Pajot showed this quasi-isometry rigidity for some buildings ([3]). The class of Tits buildings they study are called right-angled Fuchsian buildings: their apartments are hyperbolic planes, their chambers are regular hyperbolic p-gons with right angles. These buildings are $CAT(-1)$-spaces. By studying the quasi-conformal structure of the boundary at infinity of those buildings, Bourdon and Pajot showed the following quasi-isometry rigidity:

Theorem 1. *Let Δ, Δ' be two right-angled Fuchsian buildings. Any quasi-isometry $F : \Delta \to \Delta'$ lies within bounded distance from an isometry.*

This theorem was later proved for general Fuchsian buildings by Xie [8] (see section 1.5).

Let us briefly explain the quasi-conformal analysis on the Fuchsian buildings investigated by Bourdon-Pajot and Xie. As Fuchsian buildings are Gromov hyperbolic, their boundaries $(\partial\Delta, \delta)$ with a visual metric are Ahlfors Q-regular (see lectures of T. Dymarz and those of Y. Kim). (There is an optimal Q for which the space is Q-regular.) On the other hand, we already learned that a complete Q-regular metric space enjoys Q-Loewner property iff $(1, Q)$-Poincare inequality holds (lecture of H. Hakobyan) , and vaguely speaking it is equivalent to showing that there are many rectifiable curves (lecture of I. Peng). One equivalent condition that we have not seen yet is that for a complete Q-regular space, having a $(1, Q)$-Poincare inequality is equivalent to the fact that the given metric attains the conformal dimension

$$Cdim(\partial X) = \inf\{Hdim(d) : d \text{ quasi} - \text{conformal to } \delta\},$$

where $Hdim(d)$ denotes the Hausdorff dimension of $(\partial X, d)$. The necessary condition is due to Bonk-Tyson, for Ahlfors Q-regular metric spaces with non-trivial Q-modulus. The sufficient condition is due to Keith-Laakso, who showed that attaining conformal dimension (with $Cdim = Q$) implies that some pointed Gromov-Hausdorff limit space has Q-Loewner property. In our case of boundary of Gromov hyperbolic spaces, this limit is the space itself. One sufficient condition of attaining conformal dimension is that the space has some "product structure" (product of two spaces of dimension 1 and $Hdim(\delta) - 1$, corresponding to curves and its fibers). This is an important lemma by Pansu [7], and it is used by Bourdon [2] to show that the "combinatorial metric" he constructed attains the conformal dimension, thus the space enjoys Loewner property. Roughly speaking, he showed that the boundary of a right-angled building is a "product" (à la Pansu) of a circle and a Cantor set (the boundary of a hyperbolic plane and the boundary of a tree). In terms of "finding many rectifiable curves", he showed that the orbits of a geodesic segment γ in $\partial\mathbb{H}^2$ under the group of isometries of Δ fixing certain chamber and endpoints of γ, form a nice set of curves.

1.2 Preliminaries

1.2.1 $(\mathbf{p},\mathbf{q})$-Fuchsian buildings

Building Δ: In this section, we define (p,q)-Fuchsian buildings ([2]). These are 2-dimensional right-angled hyperbolic buildings with thickness q. See [5] for the definition of general hyperbolic buildings. Let us give three descriptions of the building.

(i) Let P be a compact, convex regular polygon with p edges, say $e_1,\cdots,e_p$, in $\mathbb{H}^2$ with all dihedral angles $\pi/2$. Locally the building Δ is constructed by gluing copies of polygon P as follows. Start with one copy. Attach $q-1$ copies along each edge. Around a fixed vertex, say intersection of edge e_i and e_{i+1}, there are two types of edges (i.e. they are copies of either e_i or e_{i+1}) since the dihedral angle is $\pi/2$. Now for any edge of type e_i and any edge of type e_{i+1}, we glue another copy of P along those two edges. (This amounts to saying that the link of each vertex is a complete (q,q) bipartite graph.)

(ii) The building Δ is the universal cover of the complex of groups on P with the following local groups and obvious monomorphisms: The local group of P is trivial, all the edge group are cyclic groups C of order q, and all the vertex groups are the direct product $C\times C$. The fundamental group of the complex of groups is

$$W=< s_1,\cdots,s_p : s_i^q=1,[s_i,s_{i+1}]=1 > .$$

(iii) Here is a precise definition. Let P be as above, and let (W,I) be the right-angled Coxeter system generated by reflections in the edges of P. A (p,q)-Fuchsian building Δ is a polyhedral complex with a maximal family of subcomplexes (called *apartments*), each isometric to the tesselation of $\mathbb{H}^2$ by copies of P (called *chambers*), which satisfy the usual axioms of buildings:

1. For any two chambers, there exists an apartment containing them.
2. For any two apartments A, A' with non-trivial intersection, there exists an isometry $A\to A'$ fixing the intersection $A\cap A'$ pointwise.

Retractions: For a fixed chamber c and an apartment A containing it, there exists a map $\rho:\Delta\to A$, called the *retraction* from Δ onto A centered at c. It fixes c pointwise, and its restriction to any apartment A' is an isometry fixing $A\cap A'$.

The boundary $\partial\Delta$: The building Δ is a Gromov hyperbolic space, thus we can define the boundary $\partial\Delta$ as the set of equivalence classes of geodesic rays (two geodesic rays r, r' are equivalent if $\sup_t\{r(t)-r'(t)\} < \infty$). Equivalently, fix a base point x in Δ and define $\partial\Delta$ as the set of geodesic rays starting from x. With the topology of uniform convergence on compact subsets of $[0,\infty)$, the boundary $\partial\Delta$ is homeomorphic to the Menger sponge ([1]).

Tree walls: A *wall* of Δ is a bi-infinite geodesic contained in the 1-skeleton of Δ. Since the dihedral angle is $\pi/2$, all the edges of a wall have the same type (i.e. they are copies of one edge of P), called *type* of the wall. Let us define an equivalence relation: two edges are equivalent if they are contained in a wall of Δ. An equivalence class is called a *tree-wall* and its *type* is the type of the edges in it. A tree-wall of type i is a totally geodesic, bi-homogeneous tree which divides Δ into q connected components. Since they are totally geodesic, any two distinct tree-walls share at most one vertex of Δ. Also, a geodesic intersects a tree-wall at exactly one point if it does transversally. Let us denote the set of tree-walls by $\mathcal{T}$.

1.2.2 Combinatorial metrics

In this section, we describe metric structures on the boundary $\partial\Delta$ of a (p,q) Fuchsian building Δ. Let us first define a metric $|\cdot-\cdot|$ on the set of chambers of Δ (or on the dual graph of the one skeleton of Δ), which is the path metric determined by letting $|c-d|=1$, if c,d are adjacent chambers with common edge e_i. For any pair of chambers c,d of Δ and a tree wall T, let $\alpha_T(c,d)=1$ if c,d belong to two distinct connected components of $(\Delta\cup\partial\Delta)\backslash(T\cup\partial T)$, and zero otherwise. Then

$$|c-d| = \sum_{T\in\mathcal{T}} \alpha_T(c,d).$$

Note that there are only finitely many T's which seperates c and d (exactly those that intersect the geodesic segment, in the dual graph, joining c and d). For chambers c,d,e of Δ, and $\xi\in\partial\Delta$, we define the *combinatorial horospherical distance* as follows:

$$N_\xi(c,d) = \sum_{T\in\mathcal{T}} \alpha_T(c,\xi)-\alpha_T(d,\xi).$$

For chambers c, d, e of Δ, their Gromov product is defined as

$$\{d|e\}_c = \frac{1}{2}\{|c-d| + |c-e| - |d-e|\}.$$

For any chamber c and any pair d, e of chambers, there are only finitely many tree walls seperating c and $\{d, e\}$. By connectivity argument, for any pair ξ, ν of poins on the boundary $\partial^2\Delta = (\partial\Delta)^2 \backslash \mathrm{diag}(\Delta)$, the *combinatorial Gromov product with base point* c

$$\{\xi|\nu\}_c = \frac{1}{2}\sum_{T\in\mathcal{T}} \alpha_T(c,d) + \alpha_T(c,e) - \alpha_T(d,e)$$

is well-defined.

Remark. In more general case when the thickness q_i ($q-1$ in the above case) depends on the edge e_i, we define $|c-d| = \log q_i$. Note that this metric is not a word metric, and it is not "symmetric" in the sense that the length of your step (in the dual graph of the 1-skeleton of the building) depends on the "direction" i.e. on the edge you cross).

The combinatorial metric: This is a length metric on $\partial\Delta$ induced by the metric $|\cdot - \cdot|$ as follows. Let τ be the exponential growth rate of the dual graph of one chamber A with the metric $|\cdot - \cdot|$:

$$\tau = \limsup_{n\to\infty}\left(\frac{1}{n}\log\#\{d: \text{ chamber of } A \ s.t. \ |c-d| \leq n\}\right).$$

Let $B_c(\xi, r) = \{\nu \in \partial\Delta : e_c^{-\tau\{\xi|\nu} \leq r\}$. Let us define the length $l_c(\gamma)$ of a continuous path $\gamma \subset \partial\Delta$ by

$$\lim_{r\to 0}\inf\{\sum_i r_i\},$$

where the infimum is taken over all the finite coverings $\{B_c(\xi_i, r_i)\}$ of γ with $\xi_i \in \gamma$ and $r_i < r$ for some fixed r. Now the combinatorial metric is defined as follows: for any $\xi, \nu \in \partial\Delta$,

$$\delta_c(\xi, \nu) = \inf_\gamma\{l_c(\gamma)\},$$

where the infimum is taken over all continuous paths γ in $\partial\Delta$ joining ξ and ν.

Proposition 2. *The combinatorial metric δ_c is a length metric with the following properties: (i) There is a constant $C > 1$ such that*

$$C^{-1}e^{-\tau\{\xi|\nu\}_c} \leq \delta_c(\xi,\nu) \leq Ce^{-\tau\{\xi|\nu\}_c}.$$

(ii) The combinatorial metrics are pairwise conformal: for any $\xi \in \partial\Delta$ which is not an endpoint of a wall,

$$\lim_{\nu\to\xi} \frac{\delta_d(\xi,\nu)}{\delta_c(\xi,\nu)} = e^{\tau N_\xi(c,d)}.$$

Moreover, any isometry of Δ is a conformal homeomorphism of the boundary $(\partial\Delta, \delta_c)$.

Sullivan's criterion Let H and $\mathcal{H}_c$ be the Hausdorff dimension and the Hausdorff measure of $(\partial\Delta, \delta_c)$, respectively. Using Proposition 2, it can be shown that the measure on $\partial^2\Delta$ defined by

$$\mu(\xi,\nu) = e^{2H\tau\{\xi|\nu\}_c}\mathcal{H}_c(\xi)(H)_c(\nu) \quad (*)$$

does not depend on the choice of c and is invariant under the diagonal action of the group of isometries. The following characterization of isometries of Δ is due to Sullivan for non-compact rank-1 symmetric spaces.

Proposition 3. *Let μ, μ' be the measures defined as in $(*)$. A homeomorphism $f : \partial\Delta \to \partial\Delta'$ is an extension of an isometry $\Delta \to \Delta'$ if and only if f satisfies $(f \times f)_*\mu = C\mu'$.*

1.3 Poincare inequality on the boundary

1.3.1 Approximation of curves on the boundary by geodesics in apartments

Using tree-wall structure, which is special to right-angled hyperbolic buildings, Bourdon and Pajot show that curves on the boundary of the building can be approximated by geodesic segments in the boundaries of apartments.

Lemma 4. *(i) Any two points on $\partial\Delta$ can be joined by four geodesic segments, each segment contained in the boundary of an apartment.*
(ii) Now fix a boundary point $\xi \in \partial\Delta$. For almost every $\xi \in \partial\Delta$, if $\gamma : [0,1] \to \partial\Delta$ is a continuous curve starting at ξ, then for any $t \in (0,1]$, the subcurve $\gamma((0,t])$ intersects at least one boundary of an apartment containing ξ.

1.3.2 Poincaré inequality and Loewner space structure

Using the above lemma and the uniform Ahlfors-regularity of the fibers of the retraction ρ (of Δ onto A), we can estimate the size of the *pencil joining two points* on the boundary of A (this is the set of boundaries of apartments containing both points). Poincaré inequalities follows from these estimates:

Proposition 5. *([4]) The metric space* $(\partial\Delta, \delta)$ *admits weak* $(1, \alpha)$*-Poincaré inquality for every* $\alpha \geq 1$.

Existence of Poincaré inequality implies absolute continuity of quasi-symmetric homeomorphisms and Loewner space structure of the boundary $(\partial\Delta, \delta)$.

Remark. Bourdon proved earlier ([2]) that the combinatorial metric he constructed satisfies the "product structure" needed to apply Pansu's lemma. Note that only the $(1, Q)$-Poincaré inequality for $Q = Cdim > 1$ is implied by Pansu's Lemma. Bourdon and Pajot ([4]) showed $(1, 1)$-Poincaré inequality by proving it directly (using the same set of curves mentioned in section 1.1).

1.4 Proof of theorem

By Gromov's theorem, any quasi-isometry $\Delta \to \Delta'$ induces a quasi-symmetric homeomorphism on the boundaries $\partial\Delta \to \partial\Delta'$. The main theorem follows form the following:

(i) this quasi-symmetric homeomorphism is in fact conformal,

(ii) any conformal map on the boundaries is induced from an isometry of the buildings.

1.4.1 Conformal homeomorphism

To show that any quasi-symmetric homeomorphism $f : \partial\Delta \to \partial\Delta'$ is conformal, we need differentiability property of such homeomorphisms and a version of Rademacher-Stepanov theorem.

For any boundary $a = \partial A$ of an apartment, let $a(t)$ be the arc-length parameterization of a s.t. $a(0) = \xi$. For $\xi \in \partial\Delta$, and $t \in (-d, d)$, set

$$D_{\xi,t}(a) = \frac{d(f(a(t)), f(\xi))}{|t|},$$

whenever a contains ξ.

Proposition 6. *(i)For $\mathcal{H}$-almost every $\xi \in \partial\Delta$, the function $D_{\xi,t}$ converges uniformly (on the set of apartments containing ξ) to a constant function, which we denote by $f'(\xi) \in (0, \infty)$.*
(ii)For almost every $\nu \in \partial\Delta'$, we have

$$(f^{-1})(\nu) = \frac{1}{f'(f^{-1}(\nu))}.$$

From the above proposition and Lemma 4, a metric version of Rademacher-Stepanov theorem follows:

Proposition 7. *For almost every $\xi \in \partial\Delta$, we have $L_f(\xi) = f'(\xi)$.*

It follows that the ratio of quasi-symmetry $L_f(\xi)/l_f(\xi)$ equals 1 for almost every ξ.

1.4.2 A Liouville type theorem

Using Loewner property of the boundary $\partial\Delta$, it can be shown that any conformal homeomorphism $f : \partial\Delta \to \partial\Delta'$ is absolutely continuous and preserves a cross-ratio. Let M and M' be the groups of homeomorphisms of Δ and Δ', respectively, preserving the cross-ratio. It follows that the measure $(f \times f)_*\mu$ is M'-invariant in $\partial^2\Delta'$, and absolute continuity of f implies that it belongs to the class of μ'. By Sullivan's criterion (Proposition 3), f is an extension of an isometry between Δ and Δ' to their boundaries. Alternatively, it can be shown that the isometries of Δ are exactly homeomorphisms of $\partial\Delta$ which preserve extremities of walls. Preserving a cross-ratio implies this condition.

1.5 Further direction : Xie's result

A 2-dimensional hyperbolic building is called *Fuchsian* if the thickness depends only on the type of the edge. Xie extended the result of Bourdon-Pajot by showing that any homeomorphism $\partial\Delta \to \partial\Delta'$ between two Fuchsian buildings which preserves the combinatorial cross ratio almost everywhere extends to an isomorphism from Δ to Δ' ([8]). Complications arise when the polygon P has at most 4 edges. In this case, it is possible a priori to have triangles or quadrilaterals in the 1-skeleton of the building which is the union of a finite number of chambers, which does not happen in right-angled buildings. Xie showed that even when P has at most 4 edges, there are not many such triangles and quadrilaterals.

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Seonhee Lim, Cornell University
email: **slim@math.cornell.edu**