A Proof of a $L^{1/2}$ Ergodic Theorem

François Ledrappier* and Seonhee Lim
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
U.S.A.

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The aim of this note is to give a proof of the following result:

Theorem 1 *Let T be a measure preserving transformation of the probability space $(\Omega, \mathcal{A}, m)$, and $f : \Omega \mapsto \mathbb{R}$ be such that $\int \sqrt{|f|} < \infty$. Then, for m almost every $\omega \in \Omega$:*

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n=0}^{N-1} f(T^n\omega) = 0. \tag{1}$$

In the case of independent random variables with the same law in $L^{1/2}$, Theorem 1 is a case of the Marcinkiewicz-Zygmund theorem. In general actually, a stronger Theorem says that the same limit holds with identically distributed variables (not necessary stationary), see [4], Corollary page 165[1]. We want to present how Theorem 1 follows from a general Noncommutative Ergodic Theorem. We may, without loss of generality, assume that the transformation T is ergodic and invertible.

0.1 General Ergodic Theorems.

Let (X, d) be a metric space such that closed bounded sets are compact. Fix a basepoint $x_0 \in X$. Let

$$\Phi : X \to C(X)$$

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be defined by $x \mapsto d(x,\cdot) - d(x,x_0)$ and where the topology on the space of continuous functions $C(X)$ is uniform convergence on compact sets. It can be checked that Φ is a continuous injection, and we identify X with its image. Let $H = \overline{\Phi(X)}$. It is easy to verify that H is a compact and metrizable space. The points in $H \setminus \Phi(X)$ are called *horofunctions* (based at x_0).

The action by $Isom(X,d)$ on X extends continuously to an action by homeomorphisms of H and is given by

$$g.h(x) = h(g^{-1}x) - h(g^{-1}x_0).$$

See [3] and the references therein for more information.

Suppose now that $A : \Omega \mapsto Isom(X)$ is a measurable map such that $\int d(x_0, A(\omega)x_0)dm(\omega) < \infty$. Form the product

$$A^{(N)}(\omega) \;=\; A(\omega) \circ A(T\omega) \circ \cdots \circ A(T^{N-1}\omega).$$

By the Subadditive Ergodic Theorem, we know that there is a number $\alpha \geq 0$ such that, for m almost every $\omega \in \Omega$,

$$\lim_{N\to\infty} \frac{1}{N} d(x_0, A^{(N)}(\omega)x_0) \;=\; \alpha. \tag{2}$$

On the other hand, by the law of large numbers from [3], we know that, for m almost every $\omega \in \Omega$, there is a horofunction $h_\omega \in H$ such that

$$\lim_{N\to\infty} \frac{1}{N} h_\omega(A^{(N)}(\omega)x_0) \;=\; -\alpha. \tag{3}$$

0.2 Heisenberg space.

We are going to apply the above results with $X = \mathbb{R}^3$. To define the metric and the isometries, we think of X as a group with the product law defined by

$$(x,y,z)(x',y',z') \;=\; (x+x', y+y', z+z'+2(yx'-xy')).$$

Then the norm $\|(x,y,z)\| = \left(x^4+2x^2y^2+y^4+z^2\right)^{1/4}$ is subadditive ([2]). Indeed, setting $u = x+iy, u' = x'+iy'$, we can rearrange $\|(x,y,z)(x',y',z')\|^4$

as:

$$\begin{aligned}
|u|^4+z^2 &+ 4(|u|^2\operatorname{Re}u\overline{u'}+z\operatorname{Im}u\overline{u'})+\\
&+ 4(\operatorname{Re}u\overline{u'})^2+4(\operatorname{Im}u\overline{u'})^2+2|u|^2|u'|^2+2zz'+\\
&+ 4(|u'|^2\operatorname{Re}u\overline{u'}+z'\operatorname{Im}u\overline{u'})+|u'|^4+z'^2\\
\leq |u|^4+z^2 &+ 4\sqrt{|u|^4+z^2}|u||u'|+\\
&+ 6(|u|^2|u'|^2+\frac{zz'}{3})+4\sqrt{|u'|^4+z'^2}|u||u'|+|u'|^4+z'^2.
\end{aligned}$$

The sum of the last two lines is term by term not bigger than

$$\left((|u|^4+z^2)^{1/4}+(|u'|^4+z'^2)^{1/4}\right)^4,$$

which is $\left(\|(x,y,z)\|+\|(x',y',z')\|\right)^4$.

Therefore the following formula

$$d((x,y,z),(x',y',z')) \;=\; \|(x,y,z)(x',y',z')^{-1}\|$$

defines a metric on X, which is invariant under right translations. The space (X,d) is proper. By direct examination, one finds, setting $x_0=(0,0,0)$:

Proposition 2 *The space of horofunctions of (X,d) is a 2-sphere with North and South poles identified. It can be parametrized by $\{(\theta,\beta),\theta\in[-\pi,\pi),\beta\in\mathbb{R}\}\cup\{\infty\}$, as follows:*

$$h_{\theta,\beta}(x,y,z)=-\frac{x\cos\theta+y\sin\theta}{(1+\beta^2)^{3/4}}-\frac{\beta(y\cos\theta-x\sin\theta)}{(1+\beta^2)^{3/4}}$$

and $h_\infty(x,y,z)=0$. Indeed, $\Phi(u,v,w)\to h_{\theta,\beta}$ if, and only if,

$$\frac{u}{\sqrt{u^2+v^2}}\to\cos\theta,\ \frac{v}{\sqrt{u^2+v^2}}\to\sin\theta,\ \frac{w}{u^2+v^2}\to\beta,$$

and $\Phi(u,v,w)\to 0$ if, and only if, $\frac{|w|}{u^2+v^2}\to\infty$.

0.3 Proof of Theorem 1.

Let $A:\Omega\mapsto Isom(X)$ be such that $A(\omega)$ is the right translation by $(1,1,f(\omega))$. We have

$$\int d(x_0,A(\omega)x_0)dm(\omega) \;=\; \int(4+f^2)^{1/4} \;<\infty.$$

Then, $A^{(N)}(\omega)$ is the right translation by $\left(N, N, \sum_{n=0}^{N-1} f(T^n\omega)\right)$. Set $S_N(\omega) = \sum_{n=0}^{N-1} f(T^n\omega)$. By (2), there is a number α such that, for m almost every $\omega \in \Omega$,

$$\lim_N \frac{1}{N}\left(4N^4 + S_N(\omega)^2\right)^{1/4} \;=\; \alpha.$$

On the other hand, by (3) and Proposition 2, for m almost every $\omega \in \Omega$, there is $\theta(\omega), \beta(\omega)$ such that:

$$-\frac{(\cos\theta(\omega) + \sin\theta(\omega))}{(1+\beta(\omega)^2)^{3/4}} - \frac{\beta(\omega)(\cos\theta(\omega) - \sin\theta(\omega))}{(1+\beta(\omega)^2)^{3/4}} = -\alpha.$$

The horofunction h_∞ is impossible since, from the first equation, $\alpha \geq \sqrt{2} > 0$. The second equation yields that there are θ and β such that

$$\alpha \;=\; \sqrt{2}\left(\frac{\cos(\theta - \pi/4)}{(1+\beta^2)^{3/4}} - \frac{\beta\sin(\theta - \pi/4)}{(1+\beta^2)^{3/4}}\right).$$

Set $\beta = \tan\varphi$, for some $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $\alpha = \sqrt{2\cos\varphi}\cos(\theta + \varphi - \frac{\pi}{4})$. Since $\alpha \geq \sqrt{2}$, this is possible only if $\alpha = \sqrt{2}$ (and then $\theta = \frac{\pi}{4}, \varphi = 0$). Therefore, for m almost every $\omega \in \Omega$, $\lim_N \frac{1}{N}\left(4N^4 + S_N(\omega)^2\right)^{1/4} \;=\; \sqrt{2}$ and the term $\frac{1}{N^4}S_N^2$ does not contribute to the limit. This is the statement of Theorem 1.

References

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