Counting overlattices in automorphism groups of trees

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Abstract

We give an upper bound for the number $u_\Gamma(n)$ of "overlattices" in the automorphism group of a tree, containing a fixed lattice Γ with index n. For an example of Γ in the automorphism group of a $2p$-regular tree whose quotient is a loop, we obtain a lower bound of the asymptotic behavior as well.

Nous donnons une borne supérieure pour le nombre $u_\Gamma(n)$ de "surréseaux" contenant un réseau fixé d'indice n dans le groupe d'automorphismes d'un arbre. Dans le cas d'un arbre $2p$-régulier T, et d'un réseau Γ tel que $\Gamma\backslash T$ soit une boucle, nous obtenons aussi une minoration du comportement asymptotique.

Introduction. Given a connected semisimple Lie group G, the Kazhdan-Margulis lemma says that there exists a positive lower bound for the covolume of cocompact lattices in G. This is no longer true when G is the automorphism group of a locally finite tree. Bass and Kulkarni (for cocompact lattices, see [BK]) and Carbone and Rosenberg (for arbitrary lattices in uniform trees, see [CR]) even constructed examples of increasing sequences of lattices $(\Gamma_i)_{i\in\mathbb{N}}$ in $Aut(T)$ whose covolumes tend to 0 as i tends to ∞.

If Γ is a cocompact lattice in the group $Aut(T)$ of automorphisms of a locally finite tree T, there is only a finite number $u_\Gamma(n)$ of "overlattices" Γ' containing Γ with fixed index n ([B]). Thus a natural question, which was raised by Bass and Lubotzky (see [BL]), would be to find the asymptotic behavior of $u_\Gamma(n)$ as n tends to ∞.

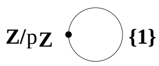
In [G], Goldschmidt proved that there are only 15 isomorphism classes of (3,3)-amalgams. Thus for lattices Γ in the automorphism group of a 3-regular tree T whose edge-indexed quotient is $\bullet\overset{3\qquad\quad 3}{\rule{4em}{0.4pt}}\bullet$, one has $u_\Gamma(n) = 0$ for n large enough. Moreover it is conjectured by Goldschmidt and Sims that there is only a finite number of (isomorphism classes of) (p,q)-almalgams, for any prime numbers p and q.

In this paper, we give two results: an upper bound of $u_\Gamma(n)$ for any cocompact lattice, and a surprisingly big lower bound of $u_\Gamma(n)$ for a specific lattice Γ in the automorphism group of a $2p$-regular tree.

Theorem 0.1. *Let Γ be a cocompact lattice in $Aut(T)$. Then there are some positive constants C_0 and C_1 depending on Γ, such that*

$$\forall n \geq 1, \qquad u_\Gamma(n) \leq C_0 n^{C_1 \log^2(n)}.$$

Theorem 0.2. *Let p be a prime number and let T be a $2p$-regular tree. Let Γ be a cocompact lattice in $Aut(T)$ such that the quotient graph of groups is a loop whose edge stabilizer is trivial and whose vertex stabilizer is a finite group of order p.*



Let $n = p_0^{k_0} p_1^{k_1} \cdots p_t^{k_t}$ be the prime decomposition of n with $p_0 = p$. Then there exist positive constants c_0, c_1 such that $\limsup_{k_0 \to \infty} \frac{u_\Gamma(n)}{n^{c_1 \log n}} \leq c_0$. For $n = p_0^{k_0} (k_0 \geq 3)$, we also have $u_\Gamma(n) \geq n^{\frac{1}{2}(k_0 - 3)}$.

It is easy to see ([B]) that

$$u_\Gamma(n) \leq \sum_{\substack{[\Gamma:\Gamma''] | n! \\ \Gamma'' \subset \Gamma}} |N_{Aut(T)}(\Gamma'')/\Gamma''|,$$

thus we could hope to use the results of Lubotzky on subgroup growth (see for instance [L1], [L2]). However, the estimations given in this way do not seem to be sharp enough. Thus our strategy consists in using the correspondence between cocompact lattices and graphs of groups (the Bass-Serre theory, see section 1) and reducing the problem to counting certain isomorphism classes of coverings of graphs of groups of index n (see section 2).

Together with the sharply contrasting examples satisfying the Goldschmidt-Sims conjecture, the examples in Theorem 0.2 are presently the only known behaviors for overlattice counting functions.

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1 Overlattices and Coverings of graphs of groups

In this section, we briefly recall some background on group actions on trees and the theory of graphs of groups, and we explain the correspondence between overlattices and coverings of graphs of groups. We refer the reader to [S], [B] and [BL] for details on the standard material, gathered in section 1.1.

Throughout the paper, we denote by T a locally finite tree, i.e., a tree having finite valence at each vertex. We denote by $Aut(T)$ the group of automorphisms without inversions of the tree T. A subgroup Γ of $Aut(T)$ is *discrete* if the stabilizer Γ_x is finite for some, thus for every, vertex x of T. The *covolume* of Γ is defined by

$$Vol\ (\Gamma\backslash\backslash T) = \sum_{x \in \Gamma\backslash VT} \frac{1}{|\Gamma_x|}.$$

A discrete subgroup is a *lattice* if its covolume is finite. In this case, $Aut(T)$ is unimodular, and the covolume is equal (up to a constant depending only on T) to the volume of $\Gamma\backslash Aut(T)$ induced by the Haar measure on the locally compact group $Aut(T)$ [BL]. A lattice Γ is called *cocompact* if the quotient graph $\Gamma\backslash T$ is finite. An *overlattice* of Γ is a lattice of $Aut(T)$ containing Γ with finite index.

1.1. Cocompact lattices and finite graphs of finite groups By a graph of groups $(X, G_\bullet)$, we mean a connected graph X, groups G_x and $G_e = G_{\overline{e}}$ assigned to each vertex x in VX and each edge e in EX, together with injections $G_e \to G_x$ for each edge e with origin $o(e) = x$. This injective map will be denoted by α_e, whatever the graph of group is. The *edge-indexed graph* of the graph of groups $(X, G_\bullet)$ is the graph X with *index* $i(e) = |G_{o(e)}|/|G_e|$ associated to each edge e. Let us denote $ad(g)(s) = gsg^{-1}$ from now on.

To every subgroup Γ of $Aut(T)$ is associated a graph of groups, well-defined up to isomorphism of graph of groups (see definition below), whose graph X is the quotient graph $\Gamma\backslash T$. We will call it a quotient graph of groups of Γ and denote it by $\Gamma\backslash\backslash T$. According to [B](section 3), a construction of $\Gamma\backslash\backslash T$ proceeds as follows. Let $p : T \to X$ be the canonical projection.

Choose subtrees $R \subset S \subset T$ such that $p|_R : R \to X$ is bijective on vertices, $p|_S : S \to X$ is bijective on edges, and for each edge e in $E(S)$, at least one of $o(e)$, $t(e)$ belongs to R. Define $\tilde{x} = p|_R^{-1}(x)$ for each x in VX and $\tilde{e} = p|_S^{-1}(e)$ with $\bar{\tilde{e}} = \tilde{\bar{e}}$ for each e in EX. For each e in EX, choose an element g_e in Γ such that $g_e o(\tilde{e}) = \widetilde{o(e)}$. We can and will always choose $g_e = 1$ for all e with $o(e) \in VR$. Note that one of $g_e, g_{\bar{e}}$ is equal to 1 for any edge e. Now let G_x be the stabilizer $\Gamma_{\tilde{x}}$ of $\tilde{x}$ in Γ for x in $VX \cup EX$. The injective map α_e is defined as $\alpha_e = ad(g_e)$. Note that for e such that $o(e)$ is a vertex of VR, each α_e is merely an inclusion.

Conversely, for any graph of groups $(X, G_\bullet)$, there exists a tree T and a group Γ acting on the tree T (unique up to equivariant tree isomorphism) such that $(X, G_\bullet)$ is isomorphic to $\Gamma\backslash\backslash T$. Let us call (T, Γ) a *universal cover* of $(X, G_\bullet)$ and Γ its *fundamental group.*

Fix $x_0 \in VX$. The fundamental group Γ of $(X, G_\bullet)$ based at x_0 is defined as follows.

The *path group* $\Pi(X, G_\bullet)$ is defined by

$$\left(\underset{x \in VX}{*} G_x \right) * F(EX)/\langle e^{-1} = \overline{e}, e\alpha_{\overline{e}}(g)e^{-1} = \alpha_e(g) : g \in G_e \rangle,$$

where $F(EX)$ denotes the free group with basis EX. For x, x' in VX, we denote by $\pi[x, x']$ the subset of $\Pi(X, G_\bullet)$ which consists of elements of the form $g_0 e_1 g_1 e_2 \cdots g_{n-1} e_n g_n$ where e_i is an edge from vertex x_{i-1} to vertex x_i, $g_i \in G_{x_i}$, $x_0 = x$, and $x_n = x'$. The *fundamental group of* $(X, G_\bullet)$ *based at* x_0 is $\Gamma = \pi_1(X, G_\bullet, x_0) = \pi[x_0, x_0]$, endowed with the group structure induced by $\Pi(X, G_\bullet)$.

The universal cover $(\widetilde{X, G_\bullet}, x_0)$ of $(X, G_\bullet)$ based at x_0 is defined as follows. It has as vertex set

$$V((\widetilde{X, G_\bullet}, x_0)) = \coprod_{x \in VX} \pi[x_0, x]/G_x,$$

and there is an edge between two distinct points $[g]$ in $\pi[x_0, x]/G_x$ and $[g']$ in $\pi[x_0, x']/G_{x'}$ if and only if $g^{-1}g' \in G_x e G_{x'}$ where e is an edge in X from x to x'. The fundamental group $\pi_1(X, G_\bullet, x_0) = \pi[x_0, x_0]$ acts on $(\widetilde{X, G_\bullet}, x_0)$ by the natural left action. The graph $(\widetilde{X, G_\bullet}, x_0)$ is a tree and moreover, for any other universal cover (T, Γ) of $(X, G_\bullet)$, there is an isomorphism ψ between Γ and $\pi_1(X, G_\bullet, x_0)$ and a ψ-equivariant graph isomorphism between T and $(\widetilde{X, G_\bullet}, x_0)$, see for example [S].

A graph of groups is called *faithful (or effective)* if there is no edge subgroup family $(N_e)_{e \in EX}$ satisfying the following conditions:

i) for each e and e' in EX such that $o(e) = o(e')$, the images of N_e and $N_{e'}$ coincide: $\alpha_e(N_e) = \alpha_{e'}(N_{e'})$. Let us denote it by $N_{o(e)}$.

ii) For each x in VX, N_x is a nontrivial normal subgroup in G_x.

It is shown in [B] that the graph of groups $(X, G_\bullet)$ is faithful if and only if its fundamental group Γ is a subgroup of $Aut(T)$ for its universal cover T, i.e., if and only if the map $\Gamma \longrightarrow Aut(T)$ is injective. The fundamental group of a faithful finite graph of finite groups is a cocompact lattice in the automorphism group of its universal covering tree and conversely, a quotient graph of groups of a cocompact lattice in the automorphism group of a locally finite tree is a faithful finite graph of finite groups.

In [B], Bass defines a covering of graphs of groups in such a way that the induced map between the corresponding fundamental groups is a group monomorphism.

Definition 1.1. *Let $(X, G_\bullet)$ and $(Y, H_\bullet)$ be two graphs of groups. We call a morphism of graphs of groups, which we denote by $\phi_\bullet = (\phi, \phi_x, \gamma_x) : (X, G_\bullet) \to (Y, H_\bullet)$, the following data*

(i) a graph morphism $\phi : X \to Y$,

(ii) group homomorphisms $\phi_x : G_x \to H_{\phi(x)}$ and $\phi_e : G_e \to H_{\phi(e)}$, for every vertex x and every edge e of X,

(iii) families of elements $(\gamma_x)_{x\in VX} \in \pi_1(Y, H_\bullet, \phi(x))$ and $(\gamma_e)_{e\in EX} \in \Pi(Y, H_\bullet)$

such that for every edge e of X with origin x, we have $\gamma_x^{-1}\gamma_e \in H_{\phi(x)}$ and the following diagram commutes.

$$\begin{tikzcd} G_e \arrow[r, "\alpha_e"] \arrow[d, "\phi_e"] & G_x \arrow[d, "\phi_x"] \\ H_{\phi(e)} \arrow[r, "ad(\gamma_x^{-1}\gamma_e)\circ\alpha_{\phi(e)}"] & H_{\phi(x)} \end{tikzcd}$$

The *induced homomorphism of path groups* $\Phi = \Phi_{\phi_\bullet} : \Pi(X, G_\bullet) \to \Pi(Y, H_\bullet)$, is defined as follows on generators (see [B]): $\Phi(g) = \gamma_x\phi_x(g)\gamma_x^{-1}$ for $g \in G_x$ and $x \in VX$, $\Phi(e) = \gamma_e\phi(e)\gamma_{\bar{e}}^{-1}$ for $e \in EX$. The induced homomorphism on path groups restricts to a homomorphism $\pi_1(X, G_\bullet, x_0) \to \pi_1(Y, H_\bullet, \phi(x_0))$, which we will denote again by Φ.

The induced homomorphism $\Phi = \Phi_{\phi_\bullet} : \pi_1(X, G_\bullet, x_0) \to \pi_1(Y, H_\bullet, \phi(x_0))$ gives a Φ_{x_o}-equivariant graph isomorphism $\tilde{\phi} : \widetilde{(X, G_\bullet, x_o)} \to \widetilde{(Y, H_\bullet, \phi(x_0))}$ defined by

$$[g] \in \pi[x_0, x]/G_x \mapsto [\Phi(g)\gamma_x] \in \pi[\phi(x_0), \phi(x)]/H_{\phi(x)}.$$

A morphism $\phi_\bullet = (\phi, \phi_x, \gamma_x)_{x\in VX\cup EX}$ of graphs of groups is an *isomorphism of graphs of groups* if ϕ is a graph isomorphism and ϕ_x are all group isomorphisms. In this case, $\phi_\bullet^{-1} = (\phi^{-1}, \phi'_y, \gamma'_y)$ where $\phi'_y = \phi_{\phi^{-1}(y)}$ and $\gamma'_y = \Phi^{-1}(\gamma_{\phi^{-1}(y)})^{-1}$ for $y \in VY \cup EY$.

Definition 1.2. *A morphism of graphs of groups $\phi_\bullet$ is furthermore called a covering if*

(a) the maps ϕ_e and ϕ_x are injective for all x and e,

(b) for every edge f of Y with origin $\phi(x)$,where x is in VX, the well-defined map

$$\Phi_{x/f} : \coprod_{e\in\phi^{-1}(f), o(e)=x} G_x/\alpha_e(G_e) \longrightarrow H_{\phi(x)}/\alpha_f(H_f)$$

$$[g]_e \longmapsto [\phi_x(g)\gamma_x^{-1}\gamma_e]_f$$

is bijective.

By the condition (b) in Definition 1.2, we have $\sum_{e\in\phi^{-1}(f), o(e)=x} \frac{|G_x|}{|G_e|} = \frac{|H_{\phi(x)}|}{|H_f|}$ for every edge f of Y with origin $\phi(x)$. Summing over all vertices x such that $\phi(x) = y$, it follows that the value of

$$n := \sum_{x\in\phi^{-1}(y)} \frac{|H_y|}{|G_x|} = \sum_{e\in\phi^{-1}(f)} \frac{|H_f|}{|G_e|}$$

does not depend on vertices and edges, since the graph Y is connected. Note that n is an integer since $\phi_x(G_x)$ is a subgroup of H_y for each x such that $\phi(x) = y$. A covering graph of groups with the above n is said to be *n-sheeted*.

Note also that by the condition (b), a covering of graphs of groups induces a covering of the corresponding edge-indexed graphs. Recall that a *covering* $\phi : (X, i) \to (Y, i)$ *of edge-indexed graphs* is a graph morphism ϕ such that $\sum_{e \in \phi^{-1}(e'), o(e)=x} i(e) = i(e')$.

Theorem 1.3 ([B], Prop. 2.7)**.** *The morphism $\phi_\bullet$ is a covering if and only if $\Phi : \pi_1(X, G_\bullet, x_0) \to \pi_1(Y, H_\bullet, \phi(x_0))$ is injective and $\widetilde{\phi} : \widetilde{(X, G_\bullet, x_0)} \to \widetilde{(Y, H_\bullet, \phi(x_0))}$ is an isomorphism.*

1.2. Counting overlattices Let Γ be a cocompact lattice in $Aut(T)$. Set

$$U(n) = U_\Gamma(n) = \{\Gamma' : \Gamma \subset \Gamma' \subset Aut(T),\ [\Gamma' : \Gamma] = n\}$$

and let $u(n) = u_\Gamma(n) = |U(n)|$ be the number of overlattices of Γ of index n. It is shown in [BK] that $u(n)$ is finite. We are interested in the asymptotic behavior of $u(n)$. For that purpose, we will show in this section that there is a bijection between overlattices of Γ and isomorphisms classes of coverings of graphs of groups by the quotient graph of groups of Γ, in the following sense.

Definition 1.4. *Let $\phi_\bullet = (\phi, \phi_x, \gamma_x) : (X, G_\bullet) \to (Y, H_\bullet)$ and $\psi_\bullet = (\psi, \psi_x, \gamma'_x) : (X, G_\bullet) \to (Y', H'_\bullet)$ be two coverings of graphs of groups. An isomorphism between them is an isomorphism of graphs of groups $\theta_\bullet = (\theta, \theta_y, \rho_y) : (Y, H_\bullet) \to (Y', H'_\bullet)$ such that $\theta \circ \phi = \psi$ as a map of graphs and the corresponding induced diagram of isomorphisms between universal covers*

$$\begin{tikzcd}
\widetilde{(X, G_\bullet, x_0)} \arrow[r, "\tilde{\phi}"] \arrow[dr, "\tilde{\psi}"'] & \widetilde{(Y, H_\bullet, \phi(x_0))} \arrow[d, "\tilde{\theta}"] \\
 & \widetilde{(Y', H', \psi(x_0))}
\end{tikzcd}$$

commutes.

It will also be useful to consider a more restricted notion of isomorphism of coverings.

Definition 1.5. *Let $\phi_\bullet = (\phi, \phi_x, \gamma_x) : (X, G_\bullet) \to (Y, H_\bullet)$ and $\psi_\bullet = (\psi, \psi_x, \gamma'_x) : (X, G_\bullet) \to (Y', H'_\bullet)$ be two coverings of graphs of groups. A strong isomorphism between them consists of a pair $\{\theta_\bullet = (\theta, \theta_y, \rho_y) : (Y, H_\bullet) \to (Y', H'_\bullet), (\zeta_x)_{x \in VX \cup EX}\}$ where $\theta_\bullet$ is an isomorphism of graphs of groups $(Y, H_\bullet) \to (Y', H'_\bullet)$ and $(\zeta_x) \in H'_{\psi(x)}$ are such that*

a) $\theta \circ \phi = \psi$ as a map of graphs,

b) For any $x \in VX \cup EX$, we have $\psi_x = ad(\zeta_x^{-1})\theta_{\phi(x)} \circ \phi_x$ as maps $G_x \to H'_{\psi(x)}$,

c) $\gamma'_x = \Theta(\gamma_x)\rho_{\phi(x)}\zeta_x$ for any $x \in VX \cup EX$.

Lemma 1.6. *Any two strongly isomorphic coverings $\phi_\bullet = (\phi, \phi_x, \gamma_x) : (X, G_\bullet) \to (Y, H_\bullet)$ and $\psi_\bullet = (\psi, \psi_x, \gamma'_x) : (X, G_\bullet) \to (Y', H'_\bullet)$ are isomorphic.*

Proof. We have a triangle of morphisms of path groups

$$\begin{tikzcd} \Pi(X, G_\bullet) \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & \Pi(Y, H_\bullet) \arrow[d, "\Theta"] \\ & \Pi(Y', H'_\bullet) \end{tikzcd}$$

We claim that this triangle commutes. It is enough to check it on generators: let $x \in VX$ and $s \in G_x$. We have $\Phi(s) = \gamma_x \phi_x(s) \gamma_x{}^{-1}$, $\Psi(s) = \gamma'_x \psi_x(s) {\gamma'_x}^{-1}$ and on the other hand

$$\begin{aligned} \Theta \circ \Phi(s) =& \Theta(\gamma_x)\Theta(\phi_x(s))\Theta(\gamma_x)^{-1} \\ =& \Theta(\gamma_x)\rho_{\phi(x)}\theta_{\phi(x)}(\phi_x(s))\rho^{-1}_{\phi(x)}\Theta(\gamma_x)^{-1} \\ =& \Theta(\gamma_x)\rho_{\phi(x)}\zeta_x\psi_x(s)\zeta_x^{-1}\rho^{-1}_{\phi(x)}\Theta(\gamma_x)^{-1} \end{aligned} \tag{1}$$

(using property (b) of strong isomorphism of coverings), and this is equal to

$$= \gamma'_x \psi_x(s) {\gamma'_x}^{-1} = \Psi(s)$$

by property (c) and the definition of Ψ. Similarly, for $e \in EX$,

$$\begin{aligned} \Theta \circ \Phi(e) =& \Theta(\gamma_e)\Theta(\phi(e))\Theta(\gamma_{\bar{e}})^{-1} \\ =& \Theta(\gamma_e)\rho_{\phi(e)}\theta(\phi(e))\rho^{-1}_{\phi(\bar{e})}\Theta(\gamma_{\bar{e}})^{-1} \\ =& \Theta(\gamma_e)\rho_{\phi(e)}\psi(e)\rho^{-1}_{\phi(\bar{e})}\Theta(\gamma_e)^{-1} = \gamma'_e\psi(e){\gamma'_{\bar{e}}}^{-1}. \end{aligned} \tag{2}$$

The last equality comes from the fact that since $\zeta_e \in H'_{\psi(e)}$, by definition of the fundamental group,

$$\psi(e) = \zeta_e \psi(e) \zeta_{\bar{e}}^{-1}.$$

Thus we have a commuting triangle of morphisms of fundamental groups

$$\begin{tikzcd} \pi_1(X, G_\bullet, x_0) \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & \pi_1(Y, H, \phi(x_0)) \arrow[d, "\Theta"] \\ & \pi_1(Y', H', \psi(x_0)) \end{tikzcd}$$

(where Θ is an isomorphism), and a triangle of *isomorphisms* of trees, which is equivariant with respect to the above triangle of groups:

$$\begin{tikzcd} (X, \widetilde{G_\bullet, x_0}) \arrow[r, "\tilde{\phi}"] \arrow[dr, "\tilde{\psi}"'] & (Y, \widetilde{H_\bullet, \phi(x_0)}) \arrow[d, "\tilde{\theta}"] \\ & (Y', \widetilde{H', \psi(x_0)}). \end{tikzcd}$$

We claim that this triangle is also commutative. Indeed, by definition, if $g \in \pi[x_0, x]/G_x \subset \widetilde{(X, G_\bullet, x_0)}$ then

$$\begin{aligned}\tilde{\theta}(\tilde{\phi}(g)) &= \tilde{\theta}(\Phi(g)\gamma_x) = \Theta(\Phi(g))\Theta(\gamma_x)\rho_{\phi(x)} \\ &= \Psi(g)\gamma'_x\zeta_x^{-1} = \Psi(g)\gamma'_x = \tilde{\psi}(g)\end{aligned}$$

where we used relation (c) together with the fact that $\zeta_x \in H'_{\psi(x)}$ (observe that $\Psi(g)\gamma'_x \in \pi[\psi(x_0), \psi(x)]/H'_{\psi(x)}$). The Lemma is proved. □

For a given overlattice Γ' of Γ, we can construct a covering $m^{\Gamma'}$ of graphs of groups as follows. Let $Y = \Gamma' \backslash T$ and $p' : T \to Y$ be the canonical projection.

Define subtrees R' and S' of R and S, respectively, in the following way. For each vertex y of Y, choose one vertex from each set $\{p'^{-1}(y)\} \cap VR$ and call it $\tilde{y}$. Let R' be the subgraph of R with vertices $\{\tilde{y} : y \in Y\}$. Since R is a tree, we can choose vertices $\tilde{y}$ so that R' is connected. Let S' be the maximal subtree of S containing R' such that $p'|_{S'}$ is injective on the edges. For $e \in EY$, choose elements $g'_e \in \Gamma'$ such that $g'_e o(\tilde{e}) = \widetilde{o(e)}$. The graph of groups $(Y, H_\bullet)$ is defined with respect to R', S' and g''s, as $(X, G_\bullet)$ is defined in section 1.1.

Now the covering of graphs of groups, which will be denoted by $m = m^{\Gamma'} : (X, G_\bullet) \to (Y, H_\bullet)$, is defined as follows. For the graph morphism $m : X \to Y$, take the natural projection π. For the group morphisms $m_x : G_x \to H_{m(x)}$, take an element σ_x in Γ' which sends $\tilde{x}$ to $\widetilde{p(x)}$. We can choose $\sigma_x = 1$ if $\tilde{x} \in VR' \cup ES'$. Note that $p(x)$ is a vertex of Y, thus $\widetilde{p(x)} \in R'$ whereas x is a vertex of X, thus $\tilde{x} \in R$. Let $m_x = ad(\sigma_x) \circ \iota$ be the injection followed by the conjugation ($g \mapsto \sigma_x g \sigma_x^{-1}$). Since G_x stabilizes $\tilde{x} \in VT \cup ET$, the group $\sigma_x G_x \sigma_x^{-1}$ stabilizes $\widetilde{p(x)} \in VT \cup ET$, thus it is a subgroup of $H_{p(x)} = \Gamma'_{\widetilde{p(x)}}$, for $x \in VX \cup EX$. For the elements γ_x, γ_e in (iii) of Definition 1.1, take $\gamma_x = \sigma_x^{-1}$ and $\gamma_e = g_e \sigma_e^{-1} g'^{-1}_{m(e)}$. It follows that

$$\begin{aligned}ad(\gamma_x^{-1}\gamma_e) \circ \alpha_{m(e)} \circ m_e &= ad(\gamma_x^{-1}\gamma_e) \circ ad(g'_{m(e)}) \circ ad(\sigma_e) \\ &= ad(\sigma_x g_e \sigma_e^{-1} g'^{-1}_{m(e)}) \circ ad(g'_{m(e)}) \circ ad(\sigma_e) \\ &= ad(\sigma_x g_e) = ad(\sigma_x) \circ ad(g_e) = m_x \circ \alpha_e.\end{aligned}$$

Since γ_x's are the elements of Γ', the map $m^{\Gamma'}$ is a morphism of graphs of groups. The maps m_x are clearly injective, thus it remains to show that the map $\Phi_{x/f}$ (in Definition 1.2. (b)) is bijective. Suppose that for $e, e' \in EX$ and $g, g' \in G_x$, we have $[\phi_x(g)\gamma_x^{-1}\gamma_e]_f = [\phi_x(g')\gamma_x^{-1}\gamma_{e'}]_f$ in $H_{\phi(x)}/\alpha_f H_f$. In other words,

$$\begin{aligned}\gamma_e^{-1}\gamma_x\phi_x(g^{-1}g')\gamma_x^{-1}\gamma_{e'} &\in \alpha_f(H_f) \\ g_{m(e)}\sigma_e g_e^{-1}\sigma_x^{-1}\sigma_x g^{-1}g'\sigma^{-1}\sigma g_{e'}\sigma_{e'}^{-1}{g'_{m(e)}}^{-1} &\in ad(g'_f)(H_f) \\ \sigma_e g_e^{-1}g^{-1}g' g_{e'}\sigma_{e'}^{-1} &\in H_f = Stab_{\Gamma'}(\tilde{f})\end{aligned}$$

Since σ_e sends $\tilde{e}$ to $\tilde{f}$ and $\sigma_{e'}$ sends $\tilde{e'}$ to $\tilde{f}$, the element $g_e^{-1}g^{-1}g'g_{e'}$ of Γ should send $\tilde{e'}$ to $\tilde{e}$. We conclude that $e = e'$ since no element of Γ sends $\tilde{e}$ to $\tilde{e'}$ where $e' \neq e$ in $X \simeq \Gamma\backslash T$. We conclude that $e = e'$ and $g^{-1}g' \in G_e$, i.e. $[g]_e = [g']_{e'}$. Therefore $m^{\Gamma'}$ is indeed a covering of graphs of groups.

Proposition 1.7. *Let Γ be a cocompact lattice of $Aut(T)$ and $(X, G_\bullet)$ be its quotient graph of groups. The map $\Gamma' \mapsto m^{\Gamma'}$ induces a bijection m between the set of overlattices of Γ of index n and the set of isomorphism classes of the n-sheeted coverings of faithful graphs of groups by $(X, G_\bullet)$.*

The following lemma shows that the map $m : \Gamma' \mapsto m^{\Gamma'}$ is well-defined.

Lemma 1.8. *Let Γ be a lattice in T, and let $\Gamma' \supset \Gamma$ be an overlattice. Fix (R, S, g_e) giving rise to a graph of groups structure $(X, G_\bullet)$ on $\Gamma\backslash T$ (as in section 1.1.). Let (R', S', g'_e) (resp. (R'', S'', g''_e)) be a data giving rise to a graph of groups structure $(Y, H_\bullet)$ (resp. $(Y', H'_\bullet)$) on $\Gamma'\backslash T$, and let $(\sigma'_x)_{x\in VX\cup EX}$ (resp. $(\sigma''_x)_{x\in VX\cup EX}$) be a data giving rise to a covering $\phi_\bullet = (\phi, \phi_x, \gamma'_x) : (X, G_\bullet) \to (Y, H_\bullet)$ (resp. $\psi_\bullet = (\psi, \psi_x, \gamma''_x) : (X, G_\bullet) \to (Y', H'_\bullet)$). Then the two coverings $\phi_\bullet$ and $\psi_\bullet$ are strongly isomorphic.*

Proof. Recall that by definition, we have $\sigma'_x : \tilde{x} \mapsto \widetilde{\phi(x)}$ and $\sigma''_x : \tilde{x} \mapsto \widetilde{\psi(x)}$, where σ'_x and σ''_x are in Γ'. Recall also that $\gamma'_x = {\sigma'_x}^{-1}, \gamma''_x = {\sigma''_x}^{-1}$ for $x \in VX$ and $\gamma'_e = g_e {\sigma'_e}^{-1} {g'_{\phi(e)}}^{-1}$, $\gamma''_e = g_e {\sigma''_e}^{-1} {g''_{\psi(e)}}^{-1}$ for $e \in EX$. Now we want to construct a strong isomorphism $\{\theta_\bullet : (Y, H_\bullet) \to (Y', H'_\bullet), \zeta_x\}$ of coverings of graphs of groups. First notice that there is a canonical bijection $\theta : Y \simeq \Gamma'\backslash T \simeq Y'$. It lifts to a bijection $\tilde{\theta} : R' \to R''$ and it extends to a unique bijection $\tilde{\theta} : S' \to S''$. Let us choose arbitrary elements $\xi_y \in \Gamma'$ for $y \in VY \cup EY$ such that $\xi_y(\tilde{y}) = \widetilde{\theta(y)}$ and define maps

$$\theta_y : H_y = \Gamma'_{\tilde{y}} \to \Gamma'_{\widetilde{\theta(y)}} = H'_{\theta(y)}, h \mapsto \xi_y h \xi_y^{-1}.$$

We have a morphism of graphs of groups $\theta_\bullet = (\theta, \theta_y, \rho_y) : (Y, H_\bullet) \to (Y', H'_\bullet)$ by setting $\rho_y = \xi_y^{-1}$ for $y \in VY$ and $\rho_e = {g'_e}^{-1} {\xi_e}^{-1} g''_{\theta(e)}$ for $e \in EY$. It is clear by construction that this is an isomorphism of graphs of groups (all maps are isomorphisms of groups). Note that there is a commutative diagram of isomorphisms

$$\begin{array}{ccc} \pi_1(Y, H_\bullet, y_0) & \xrightarrow{i_Y} & \Gamma' \\ \downarrow \Theta & & \downarrow Id \\ \pi_1(Y', H'_\bullet, \theta(y_0)) & \xrightarrow{i_{Y'}} & \Gamma' \end{array}$$

where we have denoted by i_Y, i'_Y the isomorphisms $\pi_1(Y, H_\bullet, y_0) \simeq \Gamma'$, and $\pi_1(Y', H'_\bullet, \theta(y_0)) \simeq \Gamma'$, respectively.

Finally, put $\zeta_x = \xi_x \sigma'_x {\sigma''_x}^{-1}$. For any vertex x, there holds

$$\begin{aligned} Ad(\zeta_x^{-1})\theta_{\phi(x)}\phi_x &= ad((\sigma''_x)(\sigma'_x)^{-1}\xi_{\phi(x)}^{-1}) \circ ad(\xi_{\phi(x)}) \circ ad(\sigma'_x) \\ &= ad((\sigma''_x)) = \psi_x \end{aligned}$$

as desired. A similar computation holds for $\psi_e : G_e \to H'_{\psi(e)}$ when $e \in EX$. This proves condition (b) in the definition of strong isomorphism of coverings. Condition (c) follows from the very definition of $\zeta_x, \sigma_y, \gamma'_x$ and γ''_x. $\square$

Now let us define the inverse map $\phi_\bullet \mapsto \Gamma_\phi$ of m as follows. Set $\Gamma_Y := \pi_1(Y, H_\bullet, \phi(x_0)) \subset Aut((Y, \widetilde{H_\bullet, \phi}(x_0)))$. We define an embedding $i_\phi : \Gamma_Y \to Aut((X, \widetilde{G_\bullet, x_0}))$ as follows :

$$i_\phi(u) \cdot v = \tilde{\phi}^{-1}(u \cdot \tilde{\phi}(v)) \qquad for\ u \in \Gamma_Y\ and\ v \in V(X, \widetilde{G_\bullet, x_0}) \cup E(X, \widetilde{G_\bullet, x_0}).$$

Let us denote by $\Gamma_\phi \subset Aut((X, \widetilde{G_\bullet, x_0}))$ the image of i_ϕ. The following lemma shows that this map is well-defined.

Lemma 1.9. *If $\phi_\bullet : (X, G_\bullet) \to (Y, H_\bullet)$ and $\psi_\bullet : (X, G_\bullet) \to (Y', H'_\bullet)$ are isomorphic coverings of graphs of groups, then the corresponding subgroups $\Gamma_\phi \subset Aut((X, \widetilde{G_\bullet, x_0}))$ and $\Gamma_\psi \subset Aut((X, \widetilde{G_\bullet, x_0}))$ coincide.*

Proof. By definition of isomorphic coverings, we have a triangle of *isomorphisms* of trees

$$\begin{tikzcd}
(X, \widetilde{G_\bullet, x_0}) \arrow[r, "\tilde{\phi}"] \arrow[dr, "\tilde{\psi}"'] & (Y, \widetilde{H_\bullet, \phi}(x_0)) \arrow[d, "\tilde{\theta}"] \\
 & (Y', \widetilde{H', \psi}(x_0))
\end{tikzcd}$$

which is equivariant with respect to the action of the corresponding fundamental groups. Define $\Gamma_Y \subset Aut((Y, \widetilde{H_\bullet, \phi}(x_0)))$, an embedding $i_\phi : \Gamma_Y \to Aut((X, \widetilde{G_\bullet, x_0}))$ and put $\Gamma_\phi = Im(i_\phi) \subset Aut((X, \widetilde{G_\bullet, x_0}))$ as above, and define Γ_ψ in the same fashion. We claim that $\Gamma_\phi = \Gamma_\psi$. Indeed, if $u \in \Gamma_Y$ then $\Theta(u) \in \Gamma_{Y'}$ and for $v \in (X, \widetilde{G_\bullet, x_0})$ we have

$$\begin{aligned}
i_\phi(u) \cdot v = \tilde{\phi}^{-1}(u \cdot \tilde{\phi}(v)) &= \tilde{\phi}^{-1}(\tilde{\theta}^{-1}(\Theta(u) \cdot \tilde{\theta}(\tilde{\phi}(v))) \\
&= \tilde{\psi}^{-1}(\Theta(u) \cdot \tilde{\psi}(v)) = i_\psi(\Theta(u)) \cdot v.
\end{aligned}$$

We deduce that $\Gamma_\phi \subset \Gamma_\psi$. Replacing $\theta_\bullet$ by its inverse and exchanging the roles of $\psi_\bullet$ and $\phi_\bullet$ we obtain the reverse inclusion $\Gamma_\psi \subset \Gamma_\phi$. Thus $\Gamma_\psi = \Gamma_\phi$ as desired. $\square$

Proof of Proposition 1.7. It remains to show that the map $\phi_\bullet \mapsto \Gamma_\phi$ is the inverse map of m. To see this, let $\Gamma' \supset \Gamma$ be an overlattice of Γ. The quotient graph of groups $\Gamma\backslash\backslash T = (X, G_\bullet)$ is formed relative to some datum (R, S, g_x); let us similarly choose datum (R', S', g'_x) inducing a quotient graph of groups $(Y, H_\bullet) = \Gamma'\backslash\backslash T$. Recall that by [S], §5.4, there are, for any $x_0 \in VX$ and $y_0 \in VY$, canonical isomorphisms $\Gamma \simeq \pi_1(X, G_\bullet, x_0)$, $T \simeq (X, \widetilde{G_\bullet, x_0})$ and $\Gamma' \simeq \pi_1(Y, H_\bullet, y_0)$, $T \simeq (Y, \widetilde{H_\bullet, y_0})$. Choosing furthermore some elements θ_x as in the proof of Lemma 1.8 we get a covering (see [B], Section 4.2)

$$m^{\Gamma'} : (X, G_\bullet) \to (Y, H_\bullet).$$

>From [B], Proposition 4.2, the following diagrams commute :

$$\begin{tikzcd} T \arrow[r, "id"] \arrow[d] & T \arrow[d] \\ \widetilde{(X, G_\bullet, x_0)} \arrow[r, "\widetilde{m^{\Gamma'}}"] & \widetilde{(Y, H_\bullet, y_0)} \end{tikzcd}$$

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & \Gamma' \arrow[d] \\ \pi_1(X, G_\bullet, x_0) \arrow[r, "M^{\Gamma'}"] & \pi_1(Y, H_\bullet, y_0) \end{tikzcd}$$

where we denote $M^{\Gamma'}$ the morphism of path groups induced by the covering $m^{\Gamma'}$.

In particular, the pullback of $\pi_1(Y, H_\bullet, y_0)$ via the composition of isomorphisms $T \simeq \widetilde{(X, G_\bullet, x_0)} \overset{\widetilde{m^{\Gamma'}}}{\to} \widetilde{(Y, H_\bullet, y_0)}$ is equal to Γ'. This shows that $\phi_\bullet \mapsto \Gamma_\phi$ is a left inverse of $\Gamma' \mapsto m^{\Gamma'}$.

To prove the other direction, let $\phi_\bullet : (X, G_\bullet) \to (Y, H_\bullet)$ be a covering of $(X, G_\bullet)$ and set $\Gamma' = \Gamma_\phi \subset Aut(\widetilde{(X, G_\bullet, x_0)})$. Now let $(Y', H'_\bullet)$ be the quotient graph of groups associated as in Section 1.1. to the action of Γ' on $\widetilde{(X, G_\bullet, x_0)}$, relative to some choices, and let $\psi_\bullet : (X, G_\bullet) \to (Y', H')$ be a covering constructed as in Section 1.2. By construction there is an isomorphism $\tilde{\psi} : \widetilde{(X, G_\bullet, x_0)} \xrightarrow{\sim} \widetilde{(Y', H'_\bullet, \psi(x_0))}$, equivariant with respect to an embedding $\Psi : \pi_1(X, G_\bullet, x_0) \hookrightarrow \pi_1(Y', H'_\bullet, \psi(x_0))$, and by the first part of the proof of Proposition 1.5., we have $\Gamma' = i_\psi(\pi_1(Y', H'_\bullet, \psi(x_0)))$. Thus, composing $\tilde{\psi}^{-1}$ with $\tilde{\phi}$ and Ψ^{-1} with Φ yields an isomorphism of trees $\tilde{\theta} : \widetilde{(Y', H'_\bullet, \psi(x_0))} \xrightarrow{\sim} \widetilde{(Y, H_\bullet, \phi(x_0))}$ which is equivariant with respect to an isomorphism $\Theta : \pi_1(Y', H'_\bullet, \psi(x_0)) \xrightarrow{\sim} \pi_1(Y, H_\bullet, \phi(x_0))$. At this point, we use the following Lemma:

Lemma 1.10 ([B], Prop. 4.4, Cor. 4.5.). *Let $(Z, K_\bullet)$ and $(W, J_\bullet)$ be two graphs of groups. For any isomorphism of trees $\tilde{\sigma} : \widetilde{(Z, K_\bullet, z_0)} \xrightarrow{\sim} \widetilde{(W, J_\bullet, w_0)}$ which is equvariant with respect to an isomorphism of fundamental goups $\Sigma : \pi_1(Z, K_\bullet, z_0) \xrightarrow{\sim} \pi_1(W, J_\bullet, w_0)$ there exists a (unique) isomorphism of graphs of groups $\omega_\bullet(Z, K_\bullet) \to (W, J_\bullet)$ such that $\tilde{\Sigma} = \tilde{\omega}$ and $\Sigma = \Omega$.*

Using the above Lemma, we conclude that there exists an isomorphism $\theta_\bullet : (Y', H'_\bullet) \to (Y, H_\bullet)$ making the diagram

$$\begin{tikzcd} \widetilde{(X, G_\bullet, x_0)} \arrow[r, "\tilde{\phi}"] \arrow[dr, "\tilde{\psi}"'] & \widetilde{(Y, H_\bullet, \phi(x_0))} \arrow[d, "\tilde{\theta}"] \\ & \widetilde{(Y', H', \psi(x_0))} \end{tikzcd}$$

commute. Hence the coverings $\phi_\bullet$ and $\psi_\bullet$ are indeed isomorphic as desired.

Finally, we check that the above bijection sends an overlattice of index n to an n-sheeted covering. Let Γ' be an overlattice of Γ of index n. We claim that m^Γ is an n'-sheeted covering with $n = n'$. Indeed, we have

$$n = [\Gamma' : \Gamma] = \frac{vol(\Gamma\backslash\backslash T)}{vol(\Gamma'\backslash\backslash T)} = \frac{\sum_{x\in VX}\frac{1}{|G_x|}}{\sum_{y\in VY}\frac{1}{|H_y|}} = \frac{\sum_{y\in VY}\sum_{x\in\phi^{-1}(y)}\frac{1}{|G_x|}}{\sum_{y\in VY}\frac{1}{|H_y|}} = \frac{\sum_{y\in VY}\frac{n'}{|G_y|}}{\sum_{y\in VY}\frac{1}{|H_y|}} = n'.$$

Note that the first equality comes from the fact that T is a left Γ'-set (and Γ-set) with finite stabilizers (see [BL], page 16). $\square$

It follows from the above proposition that to find $u(n)$, it suffices to count the number of isomorphism classes of coverings of faithful graphs of groups by $(X, G_\bullet)$.

2 Main results

2.1 Let G be a group of order n and let $n = \prod_{i=1}^t p_i^{k_i}$ be the prime decomposition of n. Let $\mu = \mu(n)$ be the maximum of k_i. We denote by $d(G)$ the minimal cardinality of a generating set of G and by $f(n)$ the number of isomorphism classes of groups of order n.

In [P], Pyber showed that the number of isomorphism classes of groups of order n with a given Sylow set, namely the set of Sylow p_i-subgroups defined up to conjugacy, is at most $n^{75\mu+16}$. Together with the result of Sims ([Si]), namely $f(p^k) \le p^{\frac{2}{27}k^3+\frac{1}{2}k^{\frac{8}{3}}}$, we get the following upper bound for $f(n)$:

$$\begin{aligned} f(n) &\le \prod_{i=1}^{t} p_i^{\frac{2}{27}k_i^3+\frac{1}{2}k_i^{8/3}} n^{75\mu+16} \\ &\le n^{\frac{2}{27}\mu^2+\frac{1}{2}\mu^{5/3}+75\mu+16} \end{aligned}$$

Let $g(n) = \frac{2}{27}\mu^2(n) + \frac{1}{2}\mu^{5/3}(n) + 75\mu(n) + 16$ so that $f(n) \le n^{g(n)}$.

On the other hand, Lucchini and Guralnick showed that if every Sylow subgroup of G can be generated by d elements, then $d(G) \le d + 1$ ([Luc], [Gu]). Combining with the basic fact that $d(H) \le n$ for any group H of order p^n ([Si]), we deduce that

$$d(G) \le \mu + 1.$$

Using these results, we obtain the following upper-bound for $u(n)$.

Theorem 2.1. *Let Γ be a cocompact lattice of $Aut(T)$. Then there are some positive constants C_0 and C_1 depending only on Γ, such that*

$$\forall n > 1, \qquad u_\Gamma(n) \le C_0 n^{C_1 log^2(n)}.$$

Lemma 2.2. *Any covering $\phi_\bullet = (\phi, \phi_x, \gamma_x) : (X, G_\bullet) \to (Y, H_\bullet)$ is strongly isomorphic to a covering $\phi'_\bullet = (\phi', \phi'_x, \gamma'_x) : (X, G_\bullet) \to (Y', H'_\bullet)$ where $\gamma'_x \in \Pi(Y', H'_\bullet)$ is a product of at most $12 \times$ diameter(X) generators $h_y \in H'_y$ and $e \in EY'$.*

Proof. Fix $x_0 \in X$. Associated to $\phi_\bullet$ is a lattice $\Gamma' \subset Aut(\widetilde{(X, G_\bullet, x_0)})$ containing $\pi_1(X, G_\bullet, x_0)$. From $(X, G_\bullet, x_0)$ we construct (R, S, g_e) such that the quotient of $\widetilde{(X, G_\bullet, x_0)}$ by $\pi_1(X, G_\bullet, x_0)$ is exactly $(X, G_\bullet)$. Namely, first fix a maximal tree τ in X. We may choose $R = \{e_1 \cdots e_n \mid e_1 \cdots e_n \text{ is a path from } x_0 \text{ in} \tau\}$, $S = \{e_1 \cdots e_n e_{n+1} \mid e_1 \cdots e_n \text{ is a path in } \tau\}$ and $g_e = e'_1 \cdots e'_l {e_{n+1}}^{-1} \cdots {e_1}^{-1}$ where $e'_1 \cdots e'_l$ is a path in τ from x_0 to $t(e)$, and where e is the edge connecting $e_1 \cdots e_n$ to $e_1 \cdots e_{n+1}$. In particular, g_e is a product of at most twice the diameter of X number of generators of $\Pi(X, G_\bullet)$. Now we choose R', S' subsets of R, S in such a way that the restriction of the projection $\widetilde{(X, G_\bullet, x_0)} \to \Gamma' \backslash \widetilde{(X, G_\bullet, x_0)}$ on R' is bijective for vertices (resp. the restriction of S' is bijective on edges). We also choose g'_e in a similar fashion as above, hence g'_e is also a product of at most twice the diameter of X number of generators of $\Pi(X, G_\bullet)$. From this data, we construct a graph of groups $(Y', H'_\bullet)$ as usual, and we have a canonical injection $\Gamma' \subset \Pi(Y', H'_\bullet)$. For $x \in X$ there exists a unique lift $\tilde{x} \in R$ and a unique $\tilde{x'} \in R'$ in the $\Gamma'-$orbit of $\tilde{x}$. Choose $\theta_x \in \Gamma' \subset \Pi(X, G_\bullet)$ such that $\theta_x(\tilde{x}) = \tilde{x'}$ and such that θ_x is a product of at most $l(\tilde{x}, \tilde{x_0}) + l(\tilde{x_0}, \tilde{x'})$ generators of $\Pi(X, G_\bullet)$: here $l(a, b)$ is the distance in the tree R between a and b. This is possible since we may first choose a path in the path group $\Pi(X, G_\bullet)$ from $\tilde{x}$ to $\tilde{x_0}$ of length $\leq l(\tilde{x}, \tilde{x_0})$ and then a path from $\tilde{x_0}$ to $\tilde{x'}$ of length $\leq l(\tilde{x_0}, \tilde{x'})$. Observe that since we chose $R' \subset R$, we have $l(\tilde{y}, \tilde{w}) \leq$ diameter(X) for any vertices $w, y \in VX$. We do the same thing for edges in S, to define $\theta_e \in \Pi_1(X, G_\bullet)$ such that $\theta_e(\tilde{e}) = \tilde{e'}$ and θ_e is a product of at most $2 \times$ diameter(X) generators of $\Pi(X, G_\bullet)$. Then we can construct from θ_x and θ_e's a covering $\phi'_\bullet : (X, G_\bullet) \to (Y', H'_\bullet)$, with $\gamma'_x = \theta_x^{-1}$ and $\gamma'_e = g_e \theta_e^{-1} g_{e'}^{-1}$, which are both products of at most $6 \times$ diameter(X) generators of $\Pi(X, G_\bullet)$. Observe that a word of length l in generators of $\Pi(X, G_\bullet)$ belonging to Γ' is also expressible as a word of length l in generators of $\Pi(Y', H'_\bullet)$. Finally, by the proposition on bijection of isomorphism classes of coverings and overlattices, $\phi_\bullet : (X, G_\bullet) \to (Y, H_\bullet)$ is isomorphic to $\phi'_\bullet$. □

Proof of Theorem 2.1. Let us fix a quotient graph of groups $(X, G_\bullet)$ of Γ as in section 1.1. There exist only finitely many coverings of edge-indexed graphs by the edge-indexed graphs underlying $(X, G_\bullet)$, thus it is enough to show the assertion for the number of overlattices with a fixed edge-indexed graph. Thus we want to count n-sheeted covering graphs of groups $\phi_\bullet : (X, G_\bullet) \to (Y, H_\bullet)$ such that Y is a fixed subgraph (with fixed indices) of X and $\phi : X \to Y$ the natural projection, and that the edge group H_e is a subgroup of $H_{o(e)}$.

Let $c_x = |G_x|$ for any x in $VX \cup EX$ and let $c_y = (\sum_{x \in \phi^{-1}(y)} c_x^{-1})^{-1}$. By the definition of n-sheeted covering, the cardinality $|H_y| = nc_y$, for any y in $VY \cup EY$.

Now we claim that for any group H of order n, there are at most $(m!)^{\mu(n)+1}$ subgroups of index m. For to any transitive H-action on the set $\{1, \cdots, m\}$, we can associate a subgroup of H with index m, namely the stabilizer of 1. This map $\{\rho : H \to S_m\} \longrightarrow \{H' \subset H | [H : H'] = m\}$ is surjective since for any subgroup H' of H with index m, the action of H on the cosets H/H' gives (among many) an action on $\{1, \cdots, m\}$, where we let 1 stand for the trivial coset H'. Again by the theorem of Luccini and Guralnick, there are at most $(m!)^{\mu(n)+1}$ transitive H-action on the set $\{1, \cdots, m\}$, as claimed.

There are at most $\prod_{y \in VY} (c_y n)^{g(c_y n)}$ isomorphism classes of H_y's. By the above claim, the number of subgroups $\alpha_f(H_f)$ of H_y is at most $((c_y/c_f)!)^{\mu(c_y n)+1}$. There are at most $\prod_{f \in EY} (c_f n)^{\mu(c_f n)+1}$ isomorphisms $\varphi : \alpha_f H_f \to \alpha_{\bar{f}} H_f$ and at most $\prod_{x \in VX} (c_{\phi(x)} n)^{\mu(c_x)+1}$ injections $\phi_x : G_x \to H_{\phi(x)}$. By Lemma 2.2, there are at most $(max_y |H_y|)^{12K}$ choices for each γ_x or γ_e, where $K = diameter\ of\ X$. Hence

$$\#\{(\gamma_x, \gamma_e)\} \leq \prod_{x \in VX} max_{y \in VY} (c_y n)^{12K} \times \prod_{e \in EX} max_{y \in VY} (c_y n)^{12K}$$

which is bounded by $(Mn)^{(12K)(|VX|+|EX|)}$.

Note that by the condition of injectivity and the commutativity of the diagram,

$$\begin{array}{ccc} G_e & \xrightarrow{\alpha_e} & G_x \\ \downarrow{\scriptstyle \phi_e} & & \downarrow{\scriptstyle \phi_x} \\ H_{\phi(e)} & \xrightarrow{ad(\gamma_x^{-1}\gamma_e) \circ \alpha_{\phi(e)}} & H_{\phi(x)} \end{array}$$

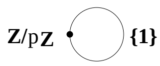
the group morphism $\phi_e : G_e \to H_\phi(e)$ is completely determined by the morphism $\phi_x : G_x \to H_x$.

Let $M = \max_{y \in VY \cup EY} c_y$, $\mu = \mu(Mn)$. Let $c_0 = |VY|$, $c_1 = |EY|$, $c_2 = \max_{\{f \in EY\}} \{(\frac{c_{o(f)}}{c_f})!\}$, let $c_3 = \sum_{x \in VX} \mu(c_x) + 1$. Combining all the estimates above, we get the following upper bound for $u(n)$,

$$\begin{aligned} u_\Gamma(n) \leq & \prod_{y \in VY} (c_y n)^{g(c_y n)} \prod_{x \in VX} (c_{\phi(x)} n)^{\mu(c_x)+1} \prod_{f \in EY} (c_f n)^{\mu(c_f n)+1} \\ & \prod_{f \in EY} ((c_{o(e)}/c_e)!)^{\mu_{c_{o(e)} n}+1} \cdot ((Mn)^{12K(|VX|+|EX|)}) \\ \leq & \prod_{y \in VY} (Mn)^{g(Mn)} \prod_{x \in VX} (Mn)^{\mu(c_x)+1} \prod_{f \in EY} (Mn)^{\mu(Mn)+1} \\ & \prod_{f \in EY} (c_2)^{\mu_{Mn}+1} \cdot ((Mn)^{12K(|VX|+|EX|)}) \\ \leq & (Mn)^{c_0 g(Mn)+c_3+c_1(\mu(Mn)+1)} (c_2)^{c_1(\mu(Mn)+1)} (Mn^{12K(|VX|+|EX|)}) \\ \leq & (Mn)^{\frac{2}{27} c_0 \mu^2 + \frac{c_0}{2} \mu^{5/3} + (75 c_0 + 2c_1)\mu + (16 c_0 + 2c_1 + c_3) c_2^{c_1(\mu+1)}} ((Mn)^{12K(|VX|+|EX|)}) \\ \leq & (C_0 n)^{C_1 \mu^2} \leq (C_0 n)^{C_1' (\log n)^2} \end{aligned}$$

where $C_0 = max\{M, c_2\}$, $C_1 = c_0\big(\frac{2}{27} + \frac{1}{2} + 75 + 16 + 2\big) + 6c_1 + c_3 + 12K(|VX| + |EX|)$ and $C_1' = \frac{C_1}{(\log 2)^2}$. $\square$

2.2. Let p be a prime number. >From now on, we assume that T is a $2p$-regular tree and that Γ is a cocompact lattice in $Aut(T)$ with a quotient graph of groups given by



The aim of this section is to give, in this situation, a smaller upper bound on $u_\Gamma(n)$ than the previous one, as well as a lower bound.

Theorem 2.3. *Let $n = p_0^{k_0} p_1^{k_1} \cdots p_t^{k_t}$ be the prime decomposition of n with $p_0 = p$. Then there exist positive constants c_0, c_1 such that $\limsup\limits_{k_0 \to \infty} \frac{u_\Gamma(n)}{n^{c_1 \log n}} \le c_0$. For $n = p_0^{k_0}$ $(k_0 \ge 3)$, we also have $u_\Gamma(n) \ge n^{\frac{1}{2}(k-3)}$.*

In the following lemma, we denote by $[g, h]$ the commutator $ghg^{-1}h^{-1}$ in G.

Lemma 2.4. *Let $A = (a_{s,t})_{1 \le s,t \le k-1}$ be a lower triangular matrix with coefficients in $0, \cdots, p-1$ and $G = G(A)$ be a group defined by the generators $\bar{g}_0, \bar{g}_1, \cdots, \bar{g}_k$ and the following relators*

$$\begin{aligned}
\bar{g}_i^p &= 1, & i &= 0, 1, \cdots, k\\
[\bar{g}_i, \bar{g}_{i+1}] &= 1, & i &= 0, 1, \cdots, k-1\\
[\bar{g}_i, \bar{g}_{i+2}] &= \bar{g}_{i+1}^{a_{1,1}}, & i &= 0, 1, \cdots, k-2\\
[\bar{g}_i, \bar{g}_{i+3}] &= \bar{g}_{i+1}^{a_{2,1}} \bar{g}_{i+2}^{a_{2,2}}, & i &= 0, 1, \cdots, k-3\\
&\vdots\\
[\bar{g}_i, \bar{g}_{i+k}] &= \bar{g}_{i+1}^{a_{k-1,1}} \bar{g}_{i+2}^{a_{k-1,2}} \cdots \bar{g}_{i+k-1}^{a_{k-1,k-1}}, & i &= 0
\end{aligned} \tag{**}$$

Then any element of G can be written as $\bar{g}_0^{i_0} \cdots \bar{g}_k^{i_k}$ where $0 \le i_j < p$.

Proof. We proceed by induction on $k \ge 1$. It is clear for the case $k = 1$ since the generators $\bar{g}_0$ and $\bar{g}_1$ commute. Now suppose that the assertion is true for all $k \le m - 1$. For $k = m$, consider the subgroup G_1 generated by $\bar{g}_0, \bar{g}_1, \cdots, \bar{g}_{m-1}$. It is a quotient of $G(A')$ with $A' = (a_{s,t})_{1 \le s,t \le k-2}$, by induction hypothesis, any element of G_1 can be written as $\bar{g}_0^{i_0} \cdots \bar{g}_{m-1}^{i_{m-1}}$ where $0 \le i_j < p$. Now we only need to consider the elements of $G - G_1$. By an easy induction, it suffices to consider the elements $\bar{g}_m \bar{g}_i = [\bar{g}_i, \bar{g}_m]^{-1} \bar{g}_i \bar{g}_m$ for $i = 0, 1, \cdots, m-1$. Since $[\bar{g}_i, \bar{g}_m] \in [G, G] \subset G_1$, the element $[\bar{g}_i, \bar{g}_m]^{-1} \bar{g}_i$ is an element in G_1, thus can be expressed as $\bar{g}_0^{i_0} \bar{g}_1^{i_1} \cdots \bar{g}_{m-1}^{i_{m-1}}$ for some i_j in $\{0, \cdots, p-1\}$. Therefore we get $\bar{g}_m \bar{g}_i = \bar{g}_0^{i_0} \cdots \bar{g}_{m-1}^{i_{m-1}} \bar{g}_m$. $\square$

Lemma 2.5. *Let G be a group of order $p^{k+1} (k \geq 1)$, G_1 and G_2 be two isomorphic subgroups of index p in G and φ be an isomorphim from G_1 to G_2. Suppose that G_1 contains no subgroup N which is normal in G and φ-invariant. Then*

(a) *there exist elements g_i in G for $i = 0, \cdots, k$, such that $\varphi(g_i) = g_{i+1}$ for $i = 0, \cdots, k-1$, $G = \langle g_0, \cdots, g_k \rangle$ and $G_1 = \langle g_0, \cdots, g_{k-1} \rangle$,*

(b) *There exists a lower triangular matrise A with coefficients in $0, \cdots, p-1$ such that the map $\psi : G(A) \to G$ defined by $\bar{g}_i \mapsto g_i$ is well-defined and is an isomorphism.*

Proof. We proceed by induction on $k \geq 1$, using the fact that any maximal proper subgroup of a p-group is normal, see for instance [Su].

We first consider the case $k = 1$, that is when G has order p^2. Since G_1 and G_2 are maximal, they are normal. Thus they are not equal by the normality assumption and $G = \langle G_1, G_2 \rangle$. Let g_0 be an element in $G_1 - G_2$ and set $g_1 = \varphi(g_0)$. Then clearly $G_1 = \langle g_0 \rangle, G_2 = \langle g_1 \rangle$ and $G = \langle g_0, g_1 \rangle$. Moreover, since $|G| = p^2$, G is abelian [Su]. Thus $[g_0, g_1] = 1$ and $G = \{g_0^{i_0} g_1^{i_1} : 0 \leq i_0, i_1 \leq p-1\}$, which shows that ψ in (b) is well-defined and surjective. Since $G(A)$ has cardinality at most p^2 by Lemma 2.3, and G has cardinality p^2, the map ψ is an isomorphism.

Now suppose that the assertion is true for all $k < m$. For $k = m$, consider G, G_1, G_2 and φ as in the statement of the lemma. As above, G_1 and G_2 are normal, distinct and $G = \langle G_1, G_2 \rangle$. Since $[G_2 : G_2 \cap G_1] \leq [\langle G_1, G_2 \rangle : G_1] = p$, we have $[G_2 : G_1 \cap G_2] = p$ and similarly $[G_1 : G_1 \cap G_2] = p$. Therefore $G_1 \cap G_2$ is maximal, thus normal in G_1 and G_2. Since G is generated by G_1 and G_2, the subgroup $G_1 \cap G_2$ is normal in G. By the assumption, $\varphi(G_1 \cap G_2) \neq G_1 \cap G_2$.

Claim. If a subgroup N of $G_1 \cap G_2$ is normal in G_2 and φ-invariant, then N is normal in G.

$$\begin{tikzcd}
& G \arrow[dl, dash] \arrow[dr, dash] & & \\
G_1 \arrow[rr, "\varphi"] \arrow[dr, dash] & & G_2 \arrow[dl, dash] \arrow[dr, dash] & \\
& G_1 \cap G_2 \arrow[rr, "\varphi"] \arrow[dr, dash] & & \varphi(G_1 \cap G_2) \arrow[dl, dash] \\
& & N &
\end{tikzcd}$$

Proof. Consider gNg^{-1}, for any $g \in G_1$. As $\varphi(gNg^{-1}) = \varphi(g)\varphi(N)\varphi(g)^{-1} = \varphi(g)N\varphi(g)^{-1} = N$ (since $\varphi(g) \in G_2$) and φ is an isomorphism, we deduce that $gNg^{-1} = \varphi^{-1}(N) = N$. Therefore $G_1 \subseteq N_G(N)$ and similarly $G_2 \subseteq N_G(N)$. Thus G, as a group generated by G_1 and G_2, is also contained in $N_G(N)$. Hence N is normal in G. $\square$

By the above claim, we can use the induction hypothesis on $G' = G_2$, $G_1' = G_1 \cap G_2$, $G_2' = \varphi(G_1 \cap G_2)$ and $\varphi' = \varphi|_{G_1 \cap G_2}$. It follows that there is an element g_1 in G_2 such that $G_2 = \langle g_1, \cdots, g_m \rangle$, $G_1 \cap G_2 = \langle g_1, \cdots, g_{m-1} \rangle$ and $\varphi(g_i) = g_{i+1}$ for $i = 1, \cdots, m-1$.

Let $g_0 = \varphi^{-1}(g_1)$. If $g_0 \in G_1 \cap G_2$, then $g_1 \in \varphi(G_1 \cap G_2)$, which contradicts $G_1 \cap G_2 \neq \varphi(G_1 \cap G_2)$. Thus g_0 is an element of $G_1 - G_2$. Since G_2 is maximal in G, the group G is generated by G_2 and g_0, i.e., $G = \langle g_0, g_1, \cdots, g_m \rangle$. It is clear that $G_1 = \langle g_0, \cdots, g_{m-1} \rangle$ as $G_1 \cap G_2$ is maximal in G_1 and $g_0 \in G_1 - (G_1 \cap G_2)$.

To prove the assertion (b), note that $[g_0, g_i] = \varphi^{-1}([g_1, g_{i+1}]) = \varphi^{-1}(g_2^{a_{i,1}} \cdots g_i^{a_{i,i}}) = g_1^{a_{i,1}} \cdots g_{i-1}^{a_{i,i}}$ for $1 \leq i \leq m-1$ and $g_i{}^p = 1$, for all $i \geq 1$ by induction hypothesis. Thus we only need to consider $[g_0, g_m]$ and $g_0{}^p$. The element g_0 clearly has order p since φ is an isomorphism and $g_1 = \varphi(g_0)$ has order p. It is easy to see that if two subgroups H and K are normal subgroups of a group G, then so is the commutator subgroup $[H, K]$ and we have $[H, K] \subset H \cap K$. Since $g_0 \in G_1$ and $g_m \in G_2$, it follows that $[g_0, g_m] \in G_1 \cap G_2 = \langle g_1, g_2, \cdots, g_{m-1} \rangle$, which proves that ψ is a well-defined homomorphism. By the previous paragraph, it is surjective. Since G and $G(A)$ are of cardinality p^{k+1} and at most p^{k+1} respectively, the map ψ is an isomorphism. $\square$

Proof of Theorem 2.2. By Proposition 1.2, the number u_n is the number of isomorphism classes of n-sheeted coverings of faithful graphs of groups $\phi_\bullet : \Gamma\backslash\backslash T \to (X, G_\bullet)$. As already seen, we may assume that $X = \Gamma\backslash T$. The following commutative diagram summarizes the data defining $\phi_\bullet$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathbb{Z}/p\mathbb{Z} \arrow[d] \\ G_e \arrow[r, shift left, "\alpha_e"] \arrow[r, shift right, "\alpha_{\overline{e}}"'] & G_x \end{tikzcd}$$

Let's first consider the case when $n = p^k$. Let $G = G_x$, $G_1 = \alpha_e(G_e)$ and $G_2 = \alpha_{\overline{e}}(G_e)$. By the condition of faithfulness, G_1 and G_2 are distinct as they are normal subgroups of G. Hence if we let $\varphi = \alpha_{\bar{e}} \circ \alpha_e^{-1} : G_1 \to G_2$, then φ is an isomorphism and there is no subgroup of G_1 which is normal in G and φ-invariant. Thus we can use Lemma 2.4 to find an element g_0 in G_x such that $G_x = \langle g_0, g_1, \cdots g_k \rangle$ where $\varphi(g_j) = g_{j+1}$. Moreover, the group G is isomorphic to $G(A)$, which is determined by A. (note that A also determines G_e and the maps α_e and $\alpha_{\bar{e}}$.) Thus we have at most $p^{\sum_{j=0}^{k-1} j}$ choices for G_x, G_e, α_e and $\alpha_{\bar{e}}$, which is exactly the number of choices of $(a_{st})_{1 \leq t \leq s \leq k-1}$. Once we have fixed G_x, G_e, α_e and $\alpha_{\bar{e}}$, an injection i from $\mathbb{Z}/p\mathbb{Z}$ into G_x is determined by the image of a generator in the domain, which implies that we have at most $|G_x| = p^{k+1}$ choices for i. Therefore we have an upper bound $u_\Gamma(n) \leq p^{\sum_{j=1}^{k-1} j} p^{k+1} = p^{\frac{(k-1)k}{2} + k + 1} = p^{\frac{k^2+k+2}{2}}$.

Now let us construct non-isomorphic classes of faithful covering graph of groups to deduce a lower bound of $u(n)$. Fix a lower triangular matrix $A = (a_{st})$ $1 \leq t \leq s \leq k-1$ with coefficients in $0, \cdots, p-1$ such that furthermore $a_{k-1,j} = 0$ for $j = 0, \cdots, k-1$. Let $G_x = G(A)$ be the group defined in Lemma 2.3, and define the covering graph of groups $\phi_\bullet = \phi_\bullet(A) : \Gamma\backslash\backslash T \to G_\bullet$ as follows. Let G_e be the subgroup of G_x generated by $\bar{g}_0, \cdots, \bar{g}_{k-1}$. Let the injection α_e be the inclusion map and the other inclusion $\alpha_{\bar{e}}$

be defined by $\alpha_{\bar{e}}(\bar{g}_i) = \bar{g}_{i+1} \in G_x$, which is indeed a monomorphism by the definition of $G(A)$. Therefore, the group morphism $\varphi = \varphi_A$ defined by $\varphi(\bar{g}_i) = \bar{g}_{i+1}$ is an isomorphism from $\alpha_e(G_e)$ onto $\alpha_{\bar{e}}(G_e)$. The data $G_\bullet$ thus defines a faithful graph of groups, as there is no φ-invariant subgroup of G_e. Indeed, for any nontrivial element $h = \bar{g}_{j0}^{i_{j0}} \cdots \bar{g}_{jt}^{i_{jt}}$ in G_e (with nonzero i_{jt}), $\varphi^{k-j_t}(h) = \bar{g}_{j0+k-j_t}^{i_{j0}} \cdots \bar{g}_k^{i_{jt}} \notin G_e$. Let v be a generator of $\mathbb{Z}/p\mathbb{Z}$ and set $\phi_x(v) = \bar{g}_0\bar{g}_k$. This defines a group monomorphism $\phi_x : \mathbb{Z}/p\mathbb{Z} \to G_x$, as $\phi_x(v)^p = (\bar{g}_0\bar{g}_k)^p = \bar{g}_0^p\bar{g}_k^p = 1$. The map ϕ_x is clearly injective since the order of g_0g_k is p. Thus we have constructed a covering of graphs of groups. Now suppose that the coverings of graphs of groups $\phi_\bullet(A)$ and $\phi_\bullet(A') : \Gamma\backslash\backslash T \to (X, G'_\bullet)$ are isomorphic. Let us denote by $\psi : (X, G_\bullet) \to (X, G'_\bullet)$ an isomorphism between them. Then there exists a commutative diagram as follows:

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] \arrow[ddr] & & \\
 & & \mathbb{Z}/p\mathbb{Z} \arrow[dl] \arrow[ddr] & \\
G_e \arrow[rr, "\psi_e"] \arrow[dr, "\alpha_e"'] & & G'_e \arrow[dr, "\alpha'_e"] & \\
 & G_x \arrow[rr, "\psi_x"'] & & G'_x
\end{tikzcd}$$

Since ψ_e (respectively $\psi_{\bar{e}}$) is a group isomorphism from $G_1 = \alpha_e(G_e)$ to $G'_1 = \alpha_e(G'_e)$ (respectively from $G_2 = \alpha_{\bar{e}}(G_e)$ to $G'_2 = \alpha_{\bar{e}}(G'_e)$), it follows that ψ_x maps the following hierarchy to the corresponding one in G'.

$$\begin{tikzcd}
 & G \arrow[dl, dash] \arrow[dr, dash] & & \\
G_1 \arrow[dr, dash] & & G_2 \arrow[dl, dash] \arrow[dr, dash] & \\
 & G_1 \cap G_2 \arrow[dr, dash] & & \varphi(G_1 \cap G_2) \arrow[dl, dash] \\
 & & \cdots &
\end{tikzcd}$$

In particular, ψ_x preserves the smallest group in the hierarchy, i.e. $\psi_x(\langle g_k \rangle) = \langle g'_k \rangle$. Let b be an element in $0, \cdots, p-1$ such that $\psi_x(g_k) = {g'_k}^b$. The map ψ_x is completely determined since $\psi_x(g_i) = {g'_i}^b$ for $i = 0, \cdots, k$. In other words, if A and bA' are not equivalent modulo p for any integer $b = 0, \cdots, p-1$, then $\phi_\bullet(A)$ and $\phi_\bullet(A')$ are non-isomorphic coverings. This implies that there are at least $\frac{p^{\sum_{i=1}^{k-2} i}}{p}$ non-isomorphic covering graphs of groups given by the above examples $\phi_\bullet(A)$ with "non-homothetic" $A = (a_{i,j})$'s. Therefore we have a lower bound $u_n \geq \frac{p^{\sum_{i=1}^{k-2} i}}{p} = p^{\frac{k^2-3k}{2}}$.

Now let's consider the general case. Recall that $|G_e| = \prod_{i=0}^t p_i^{k_i}$ and $|G_x| = \prod_{i=0}^t p_i^{k_i} p = p_0^{k_0+1} \prod_{i=1}^t p_i^{k_i}$, thus the order of the Sylow p_i-subgroup of G_e and that of G_x are the same for all $i \neq 0$. Let $G_e^{(p_i)}$ be *a* Sylow p_i-subgroup of G_e. For $i \neq 1$, let $G_x^{(p_i)} = \alpha_e(G_e^{(p_i)})$. Choose one p-Sylow subgroup $G_x^{(p)}$ of G_x containing $\alpha_e(G_e^{(p)})$.

We are now going to show that the faithfulness condition is inherited to the Sylow p-subgroups $G_e^{(p)}$, $G_x^{(p)}$ of G_e and G_x, from which we can use the upper bound given in the first part of the proof. Conjugating $\alpha_{\overline{e}}$ by an element of G_x, if necessary, we may assume that $\alpha_{\overline{e}}(G_e^{(p)}) \subset G_x^{(p)}$, thus we have the following diagram:

$$\begin{tikzcd} G_e \arrow[r, shift left, "\alpha_e"] \arrow[r, shift right, "\alpha_{\overline{e}}"'] & G_x \\ G_e^{(p)} \arrow[u, hook] \arrow[r, shift left, "\alpha_e"] \arrow[r, shift right, "\alpha_{\overline{e}}"'] & G_x^{(p)} \arrow[u, hook] \end{tikzcd}$$

Suppose that $N \lhd G_e^{(p)}$ and $\mathcal{N} = \alpha_e(N) = \alpha_{\overline{e}}(N) \lhd G_x^{(p)}$. Let $\overline{N} = \langle gNg^{-1} : g \in G_e \rangle$ (respectively $\overline{\mathcal{N}} = \langle g\mathcal{N}g^{-1} : g \in G_x \rangle$) be the smallest normal subgroup of G_e (respectively G_x) containing N (respectively $\mathcal{N}$).

Note that α_i $(i = e, \overline{e})$ induces a bijection between left cosets

$$\begin{aligned} G_e/G_e^{(p)} &\longrightarrow G_x/G_x^{(p)} \\ gG_e^{(p)} &\longmapsto \alpha_i(g)G_x^{(p)}. \end{aligned}$$

For since α_i is injective, if $gG_e^{(p)}$ is mapped to $G_x^{(p)}$, then g is in $\alpha_i(G_e) \cap G_x^{(p)}$, which is a p-group in $\alpha_i(G_e)$ containing $\alpha_i(G_e^{(p)})$. Since $\alpha_i(G_e^{(p)})$ is a Sylow p-subgroup, $\alpha_i(G_e^{(p)}) \cap G_x^{(p)}$ is equal to $\alpha_i(G_e)$. Thus $\alpha_i(g^{-1}h)$ is contained in $G_x^{(p)}$ if and only if $g^{-1}h \in G_e^{(p)}$ and the map is injective. It is surjective since the source and the target have the same cardinality. Thus any element g in G_x can be written as $\alpha_i(g')h_i$ for some $g' \in G_e$, $h_i \in G_x^{(p)}$ and we have $g\mathcal{N}g^{-1} = \alpha_i(g')h_i\mathcal{N}h_i^{-1}\alpha_i(g'^{-1}) = \alpha_i(g')\mathcal{N}\alpha_i(g'^{-1})$ $(i = 1, 2)$. Therefore $\alpha_1(\overline{N}) = \alpha_2(\overline{N}) = \overline{\mathcal{N}}$ and it is normal in G_x. As a consequence, $G_e^{(p)}$ and $G_x^{(p)}$ satisfy the condition of faithfulness, i.e., there is no subgroup N of $G_e^{(p)}$ such that $\alpha_e(N) = \alpha_{\overline{e}}(N)$ is normal in $G_x^{(p)}$. By the first part of the proof, this implies that the number of choices for $G_x^{(p)}, G_e^{(p)}, \alpha_e|_{G_e^{(p)}}$ and $\alpha_{\overline{e}}|_{G_e^{(p)}}$ is at most $p^{\frac{k_0^2+k_0+2}{2}}$.

Since all the other $G_e^{(p_i)}$ and $G_x^{(p_i)}$ have fixed cardinality, we have a constant total number of choices for them and the injections $\alpha_e|G_e^{p_i}$, say c_0. Recall that once all the $G_x^{(p_i)}$'s and $G_e^{(p_i)}$'s are chosen, the number of G_x with a given fixed Sylow system is at most $(pn)^{75\mu(pn)+16}$([P]). Recall also that the injections α_e are determined by its restriction to Sylow subgroups of G_e since they generate the group. Finally we have the following upper bound.

$$u_n \leq c_0 p^{\frac{(k_0+1)^2+(k_0+1)+2}{2}} (pn)^{75\mu(pn)+16} \leq c_0 (c_1)^{\frac{k_0^2+5k_0+8}{2}} (pn)^{75\mu+16}$$

where $c_1 = p$ and $\mu = \mu(pn)$. □

Remark. It follows from the proof above that each prime factor of $|G_x|$ is less than or equal to p, thus in the case $p = 2$, $u(n) = 0$ if n is not a power of 2.

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