Adjoint methods for the infinity Laplacian PDE

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Abstract

To study fine properties of certain smooth approximations u^ε to a viscosity solution u of the infinity Laplacian PDE, we introduce Green's function σ^ε for the linearization. We can then integrate by parts with respect to σ^ε and derive various useful integral estimates.

We are in particular able to use these estimates (i) to prove the everywhere differentiability of u and (ii) to rigorously justify interpreting the infinity Laplacian equation as a parabolic PDE.

1 Introduction

1.1 Basic equations. In this paper we consider the boundary value problem

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } U \\ u = g & \text{on } \partial U, \end{cases} \tag{1.1}$$

where $U \subseteq \mathbb{R}^n$ is an open set, $g : \partial U \to \mathbb{R}$ is Lipschitz continuous, and we write

$$\Delta_\infty u := u_{x_i} u_{x_j} u_{x_i x_j}$$

for the degenerate nonlinear *infinity-Laplacian* operator. Since the unique viscosity solution of (1.1) need not be smooth, we will study also the regularization:

$$\begin{cases} -\Delta_\infty u^\varepsilon - \varepsilon \Delta u^\varepsilon = 0 & \text{in } U \\ u^\varepsilon = g & \text{on } \partial U. \end{cases} \tag{1.2}$$

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Assume now that $V \subset\subset U$ is a compactly contained open subset, with smooth boundary. For each point $x^0 \in V$ we introduce also this linear problem:

$$\begin{cases} -(u^\varepsilon_{x_i} u^\varepsilon_{x_j} \sigma^\varepsilon)_{x_i x_j} + 2(u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} \sigma^\varepsilon)_{x_j} - \varepsilon \Delta \sigma^\varepsilon = \delta_{x^0} & \text{in } V \\ \sigma^\varepsilon = 0 & \text{on } \partial V. \end{cases} \tag{1.3}$$

Here δ_{x^0} denotes the Dirac measure at x^0.

Notation. We will write

$$L_\varepsilon v := -u^\varepsilon_{x_i} u^\varepsilon_{x_j} v_{x_i x_j} - 2 u^\varepsilon_{x_i} u^\varepsilon_{x_j x_j} v_{x_j} - \varepsilon \Delta v \tag{1.4}$$

for the linearization of (1.2), and

$$L^*_\varepsilon w := -(u^\varepsilon_{x_i} u^\varepsilon_{x_j} w)_{x_i x_j} + 2(u^\varepsilon_{x_i} u^\varepsilon_{x_j x_j} w)_{x_i} - \varepsilon \Delta w \tag{1.5}$$

for its adjoint. Thus (1.3) says

$$L^*_\varepsilon \sigma^\varepsilon = \delta_{x^0} \quad \text{in } V, \tag{1.6}$$

and so σ^ε is Green's function for the linear elliptic operator L_ε.

We will employ the functions σ^ε to extract information about the limiting behavior of u^ε as $\varepsilon \to 0$ and thus about the solution u of (1.1). The main new advances are a proof that u is everywhere differentiable and a rigorous interpretation of the infinity Laplace equation as a parabolic PDE, at least generically. Our companion paper [E-S] provides a simpler proof of the everywhere differentiability, employing only the maximum principle. This alternative proof was inspired by the adjoint methods set forth here, which however provide much more detailed information, as we will see.

Introducing the adjoint PDE (1.3) is inspired by the first author's recent paper [E1] on nonconvex Hamilton-Jacobi equations and also by various techniques for the PDE approach to weak KAM theory (see [E2]). Savin [S] proved for $n = 2$ dimensions that the viscosity solution u of (1.1) is in fact C^1.

2 Solving the approximating PDE

2.1 Estimates for u^ε. We record some first bounds, uniform in ε, proved in our other paper [E-S]:

THEOREM 2.1 (i) *There exists a unique solution u^ε of the* (1.2)*, smooth on $\bar{U}$. Furthermore, we have the estimates*

$$\max_{\bar{U}} |u^\varepsilon| \le C, \tag{2.1}$$

and for each open set $V \subset\subset U$

(2.2) $$\max_{\bar{V}} |Du^\varepsilon| \leq C.$$

Both constants are independent of ε and the constant in (2.3) *depends upon dist*$(V, \partial U) > 0$.

(ii) *We have*

(2.3) $$u^\varepsilon \to u \quad \textit{locally uniformly on } \bar{U},$$

where u is the unique viscosity solution of the boundary value problem (1.1).

2.2 The adjoint problem.

THEOREM 2.2 *There exists a unique solution σ^ε of the linear adjoint problem* (1.3), *smooth on $\bar{V} - \{x^0\}$. Furthermore,*

(2.4) $$\sigma^\varepsilon \geq 0 \quad \textit{in } V.$$

Proof. 1. According to the maximum principle, the only solution of

$$\begin{cases} L_\varepsilon v = 0 & \text{in } V \\ v = 0 & \text{on } \partial V \end{cases}$$

is $v \equiv 0$. Thus 0 is not an eigenvalue of the operator L_ε and is consequently not an eigenvalue of L_ε^*. The existence of Green's function σ^ε solving (1.3) follows from standard linear elliptic PDE theory, and σ^ε is smooth away from the singularity at x^0.

2. Given a smooth, nonnegative function f, we introduce the solution w^ε of the linear boundary value problem

(2.5) $$\begin{cases} L_\varepsilon w^\varepsilon = 0 & \text{in } V \\ w^\varepsilon = 0 & \text{on } \partial V. \end{cases}$$

Owing to the maximum principle, $w^\varepsilon \geq 0$. We multiply the PDE in (2.5) by σ^ε and integrate by parts:

$$\int_V f \sigma^\varepsilon \, dx = w^\varepsilon(x^0) \geq 0.$$

This inequality is valid for all smooth $f \geq 0$ and consequently $\sigma^\varepsilon \geq 0$. □

3 Integral identities, more estimates

3.1 A first integral identity. The following integral estimate will be useful later:

THEOREM 3.1 (i) *For each smooth function* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *we have the identity*

$$\Phi(Du^\varepsilon(x^0)) + \int_V \Phi_{p_k p_l}(Du^\varepsilon)(u^\varepsilon_{x_i} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_j} u^\varepsilon_{x_j x_l} + \varepsilon u^\varepsilon_{x_i x_l} u^\varepsilon_{x_i x_k})\sigma^\varepsilon \, dx = \int_{\partial V} \Phi(Du^\varepsilon)\rho^\varepsilon \, dS \tag{3.1}$$

for

$$\rho^\varepsilon := \left(\left(\tfrac{\partial u^\varepsilon}{\partial \nu}\right)^2 + \varepsilon \right) |D\sigma^\varepsilon|. \tag{3.2}$$

(ii) *In particular,*

$$\int_{\partial V} \rho^\varepsilon \, dS = 1 \tag{3.3}$$

and

$$Du^\varepsilon(x^0) = \int_{\partial V} Du^\varepsilon \rho^\varepsilon \, dS. \tag{3.4}$$

(iii) *We have this estimate for the second derivatives of* u^ε*:*

$$\int_V (|D^2 u^\varepsilon Du^\varepsilon|^2 + \varepsilon |D^2 u^\varepsilon|^2)\sigma^\varepsilon \, dx \le C, \tag{3.5}$$

the constant C *independent of* ε.

Observe that the density $\rho^\varepsilon = \rho^\varepsilon_{x^0}$ depends upon σ^ε and thus upon our choice of the point $x^0 \in V$. Also, take note that although (3.4) resembles a linear representation formula for $Du^\varepsilon(x^0)$, in fact ρ^ε depends in a highly nonlinear and nonlocal way upon Du^ε and $D^2 u^\varepsilon$.

Proof. 1. Differentiate the PDE (1.2) with respect to x_k:

$$L_\varepsilon u^\varepsilon_{x_k} = -u^\varepsilon_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_k x_i x_j} - 2u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} u^\varepsilon_{x_k x_j} - \varepsilon \Delta u^\varepsilon_{x_k} = 0.$$

Multiply by $\Phi_{p_k}(Du^\varepsilon)$, sum on k and rewrite, to discover that

$$L_\varepsilon \Phi = -u^\varepsilon_{x_i} u^\varepsilon_{x_j} \Phi_{x_i x_j} - 2u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} \Phi_{x_j} - \varepsilon \Delta \Phi = -\Phi_{p_k p_l}(u^\varepsilon_{x_i} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_j} u^\varepsilon_{x_j x_l} + \varepsilon u^\varepsilon_{x_i x_l} u^\varepsilon_{x_i x_k}),$$

where $\Phi = \Phi(Du^\varepsilon)$. We next multiply by σ^ε and integrate by parts twice. Recalling that $\sigma^\varepsilon = 0$ on ∂V and remembering the PDE (1.3), we discover that

$$\begin{aligned} \Phi(Du^\varepsilon(x^0)) &+ \int_V \Phi_{p_k p_l}(Du^\varepsilon)(u^\varepsilon_{x_i} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_j} u^\varepsilon_{x_j x_l} + \varepsilon u^\varepsilon_{x_i x_l} u^\varepsilon_{x_i x_k})\sigma^\varepsilon \, dx \\ &= -\int_{\partial V} \Phi(Du^\varepsilon)((u^\varepsilon_{x_i} u^\varepsilon_{x_j} \sigma^\varepsilon)_{x_i} + \varepsilon \sigma^\varepsilon_{x_j})\nu^j \, dS, \end{aligned}$$

where $\nu = (\nu^1, \dots, \nu^n)$ denotes the outward pointing unit normal along ∂V. Again noting $\sigma^\varepsilon = 0$ on ∂V, we observe that

$$\int_{\partial V} \Phi((u^\varepsilon_{x_i} u^\varepsilon_{x_j} \sigma^\varepsilon)_{x_i} + \varepsilon \sigma^\varepsilon_{x_j}) \nu^j \, dS = \int_{\partial V} \Phi(u^\varepsilon_{x_i} u^\varepsilon_{x_j} \sigma^\varepsilon_{x_i} + \varepsilon \sigma^\varepsilon_{x_j}) \nu^j \, dS = -\int_{\partial V} \Phi \rho^\varepsilon \, dS,$$

since $\nu = -\frac{D\sigma^\varepsilon}{|D\sigma^\varepsilon|}$. This proves (3.1).

2. The formulas (3.3) and (3.4) are special cases of (3.1), corresponding to $\Phi \equiv 1$ and $\Phi = p_k$ $(k = 1, \dots, n)$. The estimate (3.5) follows from the choice $\Phi = |p|^2$ and from (3.3).

□

3.2 A first estimate on the L^1 norm of σ^ε. As an application of (3.3) and (3.5), we derive a rough estimate on the integral of σ^ε:

THEOREM 3.2 *There exists a constant C such that*

$$\int_V \sigma^\varepsilon \, dx \le \frac{C}{\varepsilon^2}. \tag{3.6}$$

In general we do not have an L^1 bound for σ^ε that is independent of ε. For example, if $u = u^\varepsilon \equiv 0$, then σ^ε is $1/\varepsilon$ times Green's function for the Laplacian, in which case $||\sigma^\varepsilon||_{L^1} = O(1/\varepsilon)$. See the later Theorem 3.5 for a more refined estimate.

Proof. Let $v := |x|^2$. Then according to (1.4),

$$L_\varepsilon v := -u^\varepsilon_{x_i} u^\varepsilon_{x_j} v_{x_i x_j} - 2u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} v_{x_j} - \varepsilon \Delta v = -2(|Du^\varepsilon|^2 + n\varepsilon) - 4u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} x_j.$$

Therefore

$$\begin{aligned}
|x^0|^2 &= \int_V v L^*_\varepsilon \sigma^\varepsilon \, dx \\
&= \int_V L_\varepsilon v \sigma^\varepsilon \, dx + \int_{\partial V} |x|^2 \rho^\varepsilon \, dS \\
&= -2 \int_V (|Du^\varepsilon|^2 + n\varepsilon) \sigma^\varepsilon \, dx - 4 \int_V u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} x_j \sigma^\varepsilon \, dx + \int_{\partial V} |x|^2 \rho^\varepsilon \, dS.
\end{aligned}$$

Rearranging, we deduce that

$$\begin{aligned}
\int_V (2|Du^\varepsilon|^2 + 2n\varepsilon) \sigma^\varepsilon \, dx &\le C + C \int_U |D^2 u^\varepsilon Du^\varepsilon| \sigma^\varepsilon \, dx \\
&\le C + \frac{C}{\varepsilon} \int_V |D^2 u^\varepsilon Du^\varepsilon|^2 \sigma^\varepsilon \, dx + n\varepsilon \int_V \sigma^\varepsilon \, dx \\
&\le \frac{C}{\varepsilon} + n\varepsilon \int_V \sigma^\varepsilon \, dx,
\end{aligned}$$

according to (3.3) and (3.5). This gives (3.6). □

3.3 An exponential estimate. It is clear that when Φ is convex, the second term on the left hand side of (3.1) is nonnegative. One of our main observations is that this identity can provide useful information for certain nonconvex functions Φ, namely those of the form

$$\Phi(p) = \phi(|p|^2) \tag{3.7}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is possibly decreasing. We write $\phi = \phi(q)$.

THEOREM 3.3 (i) *For each smooth ϕ we have the identity*

$$\begin{aligned} &\phi(|Du^\varepsilon(x^0)|^2) + 2\int_V \phi'(|Du^\varepsilon|^2)(|D^2u^\varepsilon Du^\varepsilon|^2 + \varepsilon|D^2u^\varepsilon|^2)\sigma^\varepsilon\,dx \\ &\qquad = \int_{\partial V} \phi(|Du^\varepsilon|^2)\rho^\varepsilon\,dS - 4\int_V \phi''(|Du^\varepsilon|^2)((\varepsilon\Delta u^\varepsilon)^2 + \varepsilon|D^2u^\varepsilon Du^\varepsilon|^2)\sigma^\varepsilon\,dx. \end{aligned} \tag{3.8}$$

(ii) *There exists a constant $\mu > 0$ for which*

$$\int_{\partial V} \varepsilon e^{\frac{\mu(\alpha_\varepsilon^2 - |Du^\varepsilon|^2)}{\varepsilon}}\rho^\varepsilon\,dS + \int_V e^{\frac{\mu(\alpha_\varepsilon^2 - |Du^\varepsilon|^2)}{\varepsilon}}(|D^2u^\varepsilon Du^\varepsilon|^2 + \varepsilon|D^2u^\varepsilon|^2)\sigma^\varepsilon\,dx \le C\varepsilon, \tag{3.9}$$

where

$$\alpha_\varepsilon := |Du^\varepsilon(x^0)|. \tag{3.10}$$

Notice that ϕ'' occurs only within the last term in (3.8), and that this expression is $O(\varepsilon)$, according to (3.5).

We will see later that if $\liminf_{\varepsilon\to 0} |Du^\varepsilon(x^0)| > 0$, the exponential bound (3.9) implies that $u^\varepsilon(x^0)$ and $Du^\varepsilon(x^0)$ are determined up to small errors by the boundary data on ∂V only at points where $|Du^\varepsilon| \ge |Du^\varepsilon(x^0)|$. That this is should be so is suggested by our heuristic interpretation in §7 of the infinity Laplacian PDE as a parabolic equation, with "time-like" direction $-D^2uDu = -1/2D(|Du|^2)$. Therefore the values of $u(x^0)$ and $Du(x^0)$ should be determined only by boundary data "earlier in time", that is, at points on ∂V where $|Du| \ge |Du(x^0)|$.

Proof. 1. Plug the expression (3.7) into the identity (3.1), to find

$$\begin{aligned} &\phi(|Du^\varepsilon(x^0)|^2) + 2\int_V \phi'(|Du^\varepsilon|^2)(|D^2u^\varepsilon Du^\varepsilon|^2 + \varepsilon|D^2u^\varepsilon|^2)\sigma^\varepsilon\,dx \\ &\qquad = \int_{\partial V} \phi(|Du^\varepsilon|^2)\rho^\varepsilon\,dS - 4\int_V \phi''(|Du^\varepsilon|^2)((\Delta_\infty u^\varepsilon)^2 + \varepsilon|D^2u^\varepsilon Du^\varepsilon|^2)\sigma^\varepsilon\,dx \\ &\qquad = \int_{\partial V} \phi(|Du^\varepsilon|^2)\rho^\varepsilon\,dS - 4\int_V \phi''(|Du^\varepsilon|^2)((\varepsilon\Delta u^\varepsilon)^2 + \varepsilon|D^2u^\varepsilon Du^\varepsilon|^2)\sigma^\varepsilon\,dx, \end{aligned}$$

according to the PDE (1.2).

2. To establish the exponential estimate (3.9), we take

$$\phi(q) = \varepsilon e^{\frac{\mu(\alpha_\varepsilon^2 - q)}{\varepsilon}}, \tag{3.11}$$

$\mu > 0$ to be selected. Then $\phi(|Du^\varepsilon(x^0)|^2) = \varepsilon$ according to (3.10). Combining the two terms in (3.8) involving integration over V, we compute

$$\begin{aligned}
&\int_V -2\phi'(|D^2u^\varepsilon Du^\varepsilon|^2 + \varepsilon|D^2u^\varepsilon|^2)\sigma^\varepsilon - 4\phi''((\varepsilon\Delta u^\varepsilon)^2 + \varepsilon|D^2u^\varepsilon Du^\varepsilon|^2)\sigma^\varepsilon\, dx \\
&\quad = \int_V \left(2\mu(|D^2u^\varepsilon Du^\varepsilon|^2 + \varepsilon|D^2u^\varepsilon|^2) - 4\mu^2(\varepsilon(\Delta u^\varepsilon)^2 + |D^2u^\varepsilon Du^\varepsilon|^2)\right) e^{\frac{\mu(\alpha_\varepsilon^2 - |Du^\varepsilon|^2)}{\varepsilon}} \sigma^\varepsilon\, dx \\
&\quad = \int_V \left((2\mu - 4\mu^2)|D^2u^\varepsilon Du^\varepsilon|^2 + \varepsilon\mu(2|D^2u^\varepsilon|^2 - 4\mu(\Delta u^\varepsilon)^2)\right) e^{\frac{\mu(\alpha_\varepsilon^2 - |Du^\varepsilon|^2)}{\varepsilon}} \sigma^\varepsilon\, dx \\
&\quad \geq \gamma \int_V \left(|D^2u^\varepsilon Du^\varepsilon|^2 + \varepsilon|D^2u^\varepsilon|^2\right) e^{\frac{\mu(\alpha_\varepsilon^2 - |Du^\varepsilon|^2)}{\varepsilon}} \sigma^\varepsilon\, dx
\end{aligned}$$

for some positive constant γ, provided we fix $\mu > 0$ sufficiently small. □

3.4 A second integral identity. The identity (3.4) represents $Du^\varepsilon(x^0)$ as an integral of Du^ε over ∂V with respect to the density ρ^ε. We will see next that provided $|Du^\varepsilon(x^0)|$ is bounded away from zero, there is a corresponding, but approximate, formula for $u^\varepsilon(x^0)$

THEOREM 3.4 (i) *For each smooth function $\psi : \mathbb{R} \to \mathbb{R}$ we have the indentity*

$$\begin{aligned}
&\psi(u^\varepsilon(x^0)) + \int_V \psi''(u^\varepsilon)(|Du^\varepsilon|^4 + \varepsilon|Du^\varepsilon|^2)\sigma^\varepsilon\, dx \\
&\qquad = \int_{\partial V} \psi(u^\varepsilon)\rho^\varepsilon\, dS + 2\varepsilon \int_V \Delta u^\varepsilon \psi'(u^\varepsilon)\sigma^\varepsilon\, dx.
\end{aligned} \tag{3.12}$$

(ii) *If*

$$\liminf_{\varepsilon \to 0} |Du^\varepsilon(x^0)| > 0, \tag{3.13}$$

then the last term on the right of (3.12) *is* $O(\varepsilon^{\frac{1}{2}})$. *In particular,*

$$u^\varepsilon(x^0) = \int_{\partial V} u^\varepsilon \rho^\varepsilon\, dS + O(\varepsilon^{\frac{1}{2}}). \tag{3.14}$$

(iii) *Furthermore,* (3.13) *implies the estimate*

$$\int_V (|Du^\varepsilon|^4 + \varepsilon|Du^\varepsilon|^2)\sigma^\varepsilon\, dx \leq C. \tag{3.15}$$

The constant C in (3.15) depends upon a positive lower bound for $\alpha_\varepsilon := |Du^\varepsilon(x^0)|$.

Proof. 1. Multiply the PDE (1.2) by $\psi'(u^\varepsilon)$ amd rewrite, to discover that

$$-u^\varepsilon_{x_i}u^\varepsilon_{x_j}\psi_{x_ix_j} - \varepsilon\Delta\psi = -\psi''(|Du^\varepsilon|^4 + \varepsilon|Du^\varepsilon|^2),$$

where $\psi = \psi(u^\varepsilon)$. Next multiply by σ^ε and integrate by parts. Similarly to the previous proof, we learn that

$$\psi(u^\varepsilon(x^0)) + \int_V \psi''(u^\varepsilon)(|Du^\varepsilon|^4 + \varepsilon|Du^\varepsilon|^2)\sigma^\varepsilon\,dx = \int_{\partial V}\psi(u^\varepsilon)\rho^\varepsilon\,dS - 2\int_V u^\varepsilon_{x_i}u^\varepsilon_{x_ix_j}\psi_{x_j}\sigma^\varepsilon\,dx.$$

The last integral term is

$$\int_V u^\varepsilon_{x_i}u^\varepsilon_{x_ix_j}\psi_{x_j}\sigma^\varepsilon\,dx = \int_V u^\varepsilon_{x_i}u^\varepsilon_{x_j}u^\varepsilon_{x_ix_j}\psi'(u^\varepsilon)\sigma^\varepsilon\,dx = -\varepsilon\int_V \Delta u^\varepsilon\psi'(u^\varepsilon)\sigma^\varepsilon\,dx,$$

according to the PDE (1.2).

2. Our task now is to estimate the last term in (3.12), under the assumption (3.13). The main issue is that we do not yet have an L^1 estimate for σ^ε that is independent of ε.

We first consider the case that $\psi(z) = \frac{z^2}{2}$. Take ε so small that

$$\alpha_\varepsilon \geq \alpha > 0$$

for some fixed number α. The identity (3.12) for $\psi(z) = \frac{z^2}{2}$ reads

$$\frac{1}{2}(u^\varepsilon(x^0))^2 + \int_V(|Du^\varepsilon|^4 + \varepsilon|Du^\varepsilon|^2)\sigma^\varepsilon\,dx = \frac{1}{2}\int_{\partial V}(u^\varepsilon)^2\rho^\varepsilon\,dS + 2\varepsilon\int_V\Delta u^\varepsilon u^\varepsilon\sigma^\varepsilon\,dx. \tag{3.16}$$

We write the last integral as

$$\begin{aligned}2\varepsilon\int_V\Delta u^\varepsilon u^\varepsilon\sigma^\varepsilon\,dx &= 2\varepsilon\int_{V\cap\{|Du^\varepsilon|\geq\frac{\alpha}{2}\}}\Delta u^\varepsilon u^\varepsilon\sigma^\varepsilon\,dx - 2\int_{V\cap\{|Du^\varepsilon|<\frac{\alpha}{2}\}}\Delta_\infty u^\varepsilon u^\varepsilon\sigma^\varepsilon\,dx\\ &=: A + B.\end{aligned}\tag{3.17}$$

We estimate

$$\begin{aligned}|A| &\leq \varepsilon C\int_{V\cap\{|Du^\varepsilon|\geq\frac{\alpha}{2}\}}|D^2u^\varepsilon|\sigma^\varepsilon\,dx\\ &\leq \varepsilon^{\frac{1}{2}}C\left(\int_V\varepsilon|D^2u^\varepsilon|^2\sigma^\varepsilon\,dx\right)^{\frac{1}{2}}\left(\int_{V\cap\{|Du^\varepsilon|\geq\frac{\alpha}{2}\}}\sigma^\varepsilon\,dx\right)^{\frac{1}{2}}\\ &\leq \varepsilon^{\frac{1}{2}}C\left(\int_V|Du^\varepsilon|^4\sigma^\varepsilon\,dx\right)^{\frac{1}{2}}\\ &\leq \frac{1}{2}\int_V|Du^\varepsilon|^4\sigma^\varepsilon\,dx + C\varepsilon.\end{aligned}\tag{3.18}$$

The third inequality above follows from (3.5). Furthermore,

$$
\begin{aligned}
|B| &\le C \int_{V\cap\{|Du^\varepsilon|<\frac{\alpha}{2}\}} |Du^\varepsilon||D^2u^\varepsilon Du^\varepsilon|\sigma^\varepsilon\,dx \\
&\le C\left(\varepsilon\int_V |Du^\varepsilon|^2\sigma^\varepsilon\,dx\right)^{\frac12}\left(\frac{1}{\varepsilon}\int_{V\cap\{|Du^\varepsilon|<\frac{\alpha}{2}\}}|D^2u^\varepsilon Du^\varepsilon|^2\sigma^\varepsilon\,dx\right)^{\frac12} \\
&\le C\left(\varepsilon\int_V |Du^\varepsilon|^2\sigma^\varepsilon\,dx\right)^{\frac12} e^{\frac{-3\mu\alpha^2}{8\varepsilon}} \\
&\le \frac{\varepsilon}{2}\int_V |Du^\varepsilon|^2\sigma^\varepsilon\,dx + Ce^{\frac{-\gamma}{\varepsilon}},
\end{aligned}
\tag{3.19}
$$

for some $\gamma > 0$. We used the exponential estimate (3.9) for the third inequality in this calculation.

Employing the estimates (3.18) and (3.19) in (3.17) and (3.16), we derive the bound (3.15). Returning again to (3.18) and (3.19), and now using (3.15) in the next-to-last lines, we deduce that

$$|A| + |B| \le C\varepsilon^{\frac12} + Ce^{-\frac{\gamma}{\varepsilon}} = O(\varepsilon^{\frac12}).$$

This proves assertion (ii) for $\psi(z) = \frac{z^2}{2}$ and the general case follows at once from the foregoing estimates. □

3.5 An improved L^1 estimate for σ^ε. We derive next a uniform L^1 estimate for σ^ε, under the assumption that the terms $|Du^\varepsilon(x^0)|$ are bounded away from zero. This will be much more useful than the crude bound (3.6).

THEOREM 3.5 (i) *There exists a constant $\mu > 0$ such that for each $0 < \beta < \alpha_\varepsilon$, we have*

$$\int_{V\cap\{|Du^\varepsilon|\le\beta\}} \sigma^\varepsilon\,dx \le \frac{C}{\varepsilon}e^{\frac{\mu(\beta^2-\alpha_\varepsilon^2)}{\varepsilon}}, \tag{3.20}$$

where $\alpha_\varepsilon := |Du^\varepsilon(x^0)|$.

(ii) *If*

$$\liminf_{\varepsilon\to 0} |Du^\varepsilon(x^0)| > 0, \tag{3.21}$$

we have the uniform L^1 bound

$$\int_V \sigma^\varepsilon\,dx \le C. \tag{3.22}$$

Proof. 1. Let $\Phi(p) = \phi(|p|^2)$ for $\phi(q) = \varepsilon e^{\frac{\mu(\alpha_\varepsilon^2 - q)}{\varepsilon}}$ and μ the constant from the estimate (3.9).

Then $v^\varepsilon := \Phi(Du^\varepsilon)|x|^2$ satisfies

$$\begin{aligned} L_\varepsilon v^\varepsilon &= L_\varepsilon(\Phi)|x|^2 + \Phi L_\varepsilon(|x|^2) - 4u^\varepsilon_{x_i}u^\varepsilon_{x_j}\Phi_{x_i}x_j - 4\varepsilon\Phi_{x_j}x_j \\ &= -(2\phi'(|D^2u^\varepsilon Du^\varepsilon|^2 + \varepsilon|D^2u^\varepsilon|^2) + 4\phi''((\varepsilon\Delta u^\varepsilon)^2 + \varepsilon|D^2u^\varepsilon Du^\varepsilon|^2))|x|^2 \\ &\qquad - 2\Phi|Du^\varepsilon|^2 - 2n\varepsilon\Phi - 4u^\varepsilon_{x_i}u^\varepsilon_{x_j}\Phi_{x_i}x_j - 4\Phi u^\varepsilon_{x_i}u^\varepsilon_{x_jx_j}x_j - 4\varepsilon\Phi_{x_j}x_j \end{aligned}$$

where $\Phi = \Phi(Du^\varepsilon)$.

Multiplying by σ^ε and integrating, we deduce using the bound (3.9) that

$$\begin{aligned} &2\int_V (n\varepsilon + |Du^\varepsilon|^2)\Phi\sigma^\varepsilon\,dx \\ &\quad\le C\varepsilon + \int_V (-4u^\varepsilon_{x_i}u^\varepsilon_{x_j}\Phi_{x_i}x_j - 4\Phi u^\varepsilon_{x_i}u^\varepsilon_{x_jx_j}x_j - 4\varepsilon\Phi_{x_j}x_j)\sigma^\varepsilon\,dx \\ &\quad=: C\varepsilon + A_1 + A_2 + A_3. \end{aligned} \tag{3.23}$$

2. We have

$$\begin{aligned} |A_1| &\le C\int_V e^{\frac{\mu(\alpha_\varepsilon^2 - |Du^\varepsilon|^2)}{\varepsilon}}|Du^\varepsilon||\Delta_\infty u^\varepsilon|\sigma^\varepsilon\,dx \\ &= C\int_V e^{\frac{\mu(\alpha_\varepsilon^2 - |Du^\varepsilon|^2)}{\varepsilon}}|Du^\varepsilon||\varepsilon\Delta u^\varepsilon|\sigma^\varepsilon\,dx \\ &\le C\int_V e^{\frac{\mu(\alpha_\varepsilon^2 - |Du^\varepsilon|^2)}{\varepsilon}}\varepsilon(\Delta u^\varepsilon)^2\sigma^\varepsilon\,dx + \int_V |Du^\varepsilon|^2\Phi\sigma^\varepsilon\,dx \\ &\le C\varepsilon + \int_V |Du^\varepsilon|^2\Phi\sigma^\varepsilon\,dx, \end{aligned}$$

according to (3.9). We also compute

$$\begin{aligned} |A_2| &\le C\varepsilon\int_V e^{\frac{\mu(\alpha_\varepsilon^2 - |Du^\varepsilon|^2)}{\varepsilon}}|D^2u^\varepsilon Du^\varepsilon|\sigma^\varepsilon\,dx \\ &\le C\int_V e^{\frac{\mu(\alpha_\varepsilon^2 - |Du^\varepsilon|^2)}{\varepsilon}}|D^2u^\varepsilon Du^\varepsilon|^2\sigma^\varepsilon\,dx + \frac{n}{2}\int_V \varepsilon\Phi\sigma^\varepsilon\,dx \\ &\le C\varepsilon + \frac{n}{2}\int_V \varepsilon\Phi\sigma^\varepsilon\,dx, \end{aligned}$$

again according to (3.9). The estimate for A_3 is similar:

$$|A_3| \le C\varepsilon\int_V e^{\frac{\mu(\alpha_\varepsilon^2 - |Du^\varepsilon|^2)}{\varepsilon}}|D^2u^\varepsilon Du^\varepsilon|\sigma^\varepsilon\,dx \le C\varepsilon + \frac{n}{2}\int_V \varepsilon\Phi\sigma^\varepsilon\,dx.$$

We insert our estimates for A_1, A_2, A_3 into (3.23), to deduce

$$\int_V (n\varepsilon + |Du^\varepsilon|^2)\Phi\sigma^\varepsilon \, dx \leq C\varepsilon;$$

therefore

$$\varepsilon e^{\frac{\mu(\alpha_\varepsilon^2 - \beta^2)}{\varepsilon}} \int_{V\cap\{|Du^\varepsilon|\leq\beta\}} \sigma^\varepsilon \, dx \leq C.$$

This proves (3.20).

3. Assuming now (3.21), we take ε so small that

$$\alpha_\varepsilon \geq \alpha > 0$$

for some positive constant α. Then (3.20) implies for $\beta = \frac{\alpha}{2}$ that

$$\int_{V\cap\{|Du^\varepsilon|\leq\beta\}} \sigma^\varepsilon \, dx \leq Ce^{\frac{-\gamma}{\varepsilon}}$$

where $\gamma > 0$. This and (3.15) prove (3.22). □

4 Flatness estimates

In this section we assume that u is a bounded viscosity solution of the infinity Laplacian equation

$$-\Delta_\infty u = 0 \quad \text{in } B(0,3). \tag{4.1}$$

We as before introduce the regularization

$$\begin{cases} -\Delta_\infty u^\varepsilon - \varepsilon\Delta u^\varepsilon = 0 & \text{in } B(0,3) \\ \qquad\qquad u^\varepsilon = u & \text{in } \partial B(0,3). \end{cases} \tag{4.2}$$

According to Theorem 2.1,

$$\max_{B(0,2)} |u^\varepsilon|, |Du^\varepsilon| \leq C.$$

We consider also the adjoint problem on the ball $B(0,2)$:

$$\begin{cases} -(u^\varepsilon_{x_i} u^\varepsilon_{x_j} \sigma^\varepsilon)_{x_i x_j} + 2(u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} \sigma^\varepsilon)_{x_j} - \varepsilon\Delta\sigma^\varepsilon = \delta_{x^0} & \text{in } B(0,2) \\ \qquad\qquad\qquad\qquad\qquad \sigma^\varepsilon = 0 & \text{on } \partial B(0,2), \end{cases} \tag{4.3}$$

for a given point $x^0 \in B(0,1)$. As in the previous section

$$\int_{\partial B(0,2)} \rho^\varepsilon \, dx = 1 \tag{4.4}$$

for $\rho^\varepsilon = \left(\left(\frac{\partial u^\varepsilon}{\partial \nu}\right)^2 + \varepsilon \right) |D\sigma^\varepsilon|$. Furthermore, if

$$\liminf_{\varepsilon \to 0} |Du^\varepsilon(x^0)| > 0, \tag{4.5}$$

we know from Section 3 that

$$\int_{B(0,2)} (1 + |D^2 u^\varepsilon Du^\varepsilon|^2 + \varepsilon |D^2 u^\varepsilon|^2) \sigma^\varepsilon \, dx \le C, \tag{4.6}$$

and

$$\int_{B(0,2)\cap\{|Du^\varepsilon| \le \beta\}} \sigma^\varepsilon \, dx \le \frac{C}{\varepsilon} e^{\frac{\mu(\beta^2 - \alpha_\varepsilon^2)}{\varepsilon}}, \tag{4.7}$$

for some $\mu > 0$, where $\alpha_\varepsilon := |Du^\varepsilon(x^0)|$ and $0 < \beta < \alpha_\varepsilon$. The constants C in (4.6) and (4.7) depend upon a positive lower bound α for the α_ε.

In this section we make the additional "flatness" assumption that the function u^ε is uniformly close to an affine function in $B(0,2)$, which without loss we take to be the linear function x_n:

$$\max_{B(0,2)} |u^\varepsilon - x_n| =: \lambda, \tag{4.8}$$

where λ is small.

The ideal result would be that (4.8) forces the gradient Du^ε to be close to the unit vector $e_n = (0, \dots, 0, 1)$ everywhere within the ball $B(0,1)$. This however is very subtle, and we are not able to prove this. We can however show that $Du^\varepsilon(x^0)$ is close e_n, provided $x^0 \in B(0,1)$, λ is small, and $|Du^\varepsilon(x^0)|$ is close to one.

THEOREM 4.1 *Assume the condition* (4.5) *that the gradient* $Du^\varepsilon(x^0)$ *is bounded away from zero and also the flatness condition* (4.8). *Select* $x_0 \in B(0,1)$.

(i) *We then have the estimate*

$$\int_{B(0,2)} (|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2 \sigma^\varepsilon \, ds \le C\lambda. \tag{4.9}$$

(ii) *Furthermore*

$$\int_{B(0,2)\cap\{|Du^\varepsilon| \ge 1+\delta\}} \sigma^\varepsilon \, dx \le \frac{C\lambda}{\delta^2} \tag{4.10}$$

for each $\delta > 0$.

Proof. 1. Put $v^\varepsilon := (u^\varepsilon - x_n)^2$; then

$$\begin{aligned} L_\varepsilon v^\varepsilon &= -u^\varepsilon_{x_i} u^\varepsilon_{x_j} v^\varepsilon_{x_i x_j} - 2u^\varepsilon_{x_i} u^\varepsilon_{x_j x_j} v^\varepsilon_{x_j} - \varepsilon \Delta v^\varepsilon \\ &= 2(u^\varepsilon - x_n)(-u^\varepsilon_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_i x_j} - 2u^\varepsilon_{x_i} u^\varepsilon_{x_j x_j}(u^\varepsilon_{x_j} - \delta_{jn}) - \varepsilon \Delta u^\varepsilon) \\ &\quad - 2(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2 - 2\varepsilon |Du^\varepsilon - e_n|^2. \end{aligned} \tag{4.11}$$

Multiply the σ^ε and integrate over $B(0,2)$:

$$(u^\varepsilon(x^0) - x^0_n)^2 = \int_{B(0,2)} v^\varepsilon L^*_\varepsilon \sigma^\varepsilon \, dx = \int_{B(0,2)} L_\varepsilon v^\varepsilon \sigma^\varepsilon \, dx + \int_{\partial B(0,2)} v^\varepsilon \rho^\varepsilon \, dS.$$

Using (4.4), (4.8) and (4.11), we deduce that

$$\begin{aligned} \int_{B(0,2)} (|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2 \sigma^\varepsilon \, dx &\le C\lambda^2 + C\lambda \int_{B(0,2)} (|D^2 u^\varepsilon D u^\varepsilon| + \varepsilon |D^2 u^\varepsilon|) \sigma^\varepsilon \, dx \\ &\le C\lambda^2 + C\lambda, \end{aligned}$$

the last inequality a consequence of (4.6). This proves (4.9).

2. On the set $\{|Du^\varepsilon| \ge 1 + \delta\}$ we have

$$|Du^\varepsilon|^2 - u^\varepsilon_{x_n} \ge |Du^\varepsilon|(|Du^\varepsilon| - 1) \ge \delta,$$

and so (4.10) follows from (4.9). □

Next we strengthen (4.5), now to require that $|Du^\varepsilon(x^0)|$ be close to one, and then estimate by how much $Du^\varepsilon(x^0)$ differs from e_n:

THEOREM 4.2 *Select any point $x_0 \in B(0,1)$. Suppose that*

$$1 - \delta \le |Du^\varepsilon(x^0)|^2 \le 1 + \delta \tag{4.12}$$

for a small constant $\delta > 0$ and that the flatness condition (4.8) *holds.*

Then

$$|Du^\varepsilon(x^0) - e_n|^2 \le C \left(\frac{e^{\frac{-\mu\delta}{2\varepsilon}}}{\varepsilon^{\frac{1}{2}}} + \frac{\lambda^{\frac{1}{2}}}{\delta} + \lambda^{\frac{1}{4}} + \delta^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \right). \tag{4.13}$$

The conclusion (4.13) is a strong consequence of the flatness condition (4.8), since we will later be able to adjust the various parameters to make the right hand side small. But notice that we can only deduce this if we assume (4.12), that the length of the gradient is close to one.

Proof. 1. Select a smooth function ζ such that

$$\zeta \equiv 1 \text{ on } B(0,1),\ \zeta = 0 \text{ on } \partial B(0,2). \tag{4.14}$$

Then

$$\int_{B(0,2)} L_\varepsilon \zeta \, \sigma^\varepsilon \, dx = \int_{B(0,2)} \zeta \, L_\varepsilon^* \sigma^\varepsilon \, dx = \zeta(x^0) = 1. \tag{4.15}$$

We further compute that

$$\begin{aligned} L_\varepsilon(\zeta u^\varepsilon_{x_n}) &= \zeta L_\varepsilon u^\varepsilon_{x_n} + u^\varepsilon_{x_n} L_\varepsilon \zeta - 2u^\varepsilon_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_n x_i} \zeta_{x_j} - 2\varepsilon u^\varepsilon_{x_n x_i} \zeta_{x_i} \\ &= u^\varepsilon_{x_n} L_\varepsilon \zeta - 2u^\varepsilon_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_n x_i} \zeta_{x_j} - 2\varepsilon u^\varepsilon_{x_n x_i} \zeta_{x_i}, \end{aligned}$$

since our differentiating the PDE (4.2) shows $L_\varepsilon u^\varepsilon_{x_n} = 0$.

Thus (4.14) and (4.15) imply

$$\begin{aligned} u^\varepsilon_{x_n}(x^0) - 1 &= \int_{B(0,2)} (u^\varepsilon_{x_n} - 1) L_\varepsilon \zeta \sigma^\varepsilon \, dx - 2 \int_{B(0,2)} u^\varepsilon_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_n x_i} \zeta_{x_j} + \varepsilon u^\varepsilon_{x_n x_i} \zeta_{x_i} \sigma^\varepsilon \, dx \\ &=: A + B. \end{aligned} \tag{4.16}$$

2. *Estimate of A.* We recall (4.6), to compute

$$\begin{aligned} |A| &\leq \int_{B(0,2)} |u^\varepsilon_{x_n} - 1| |L_\varepsilon \zeta| \sigma^\varepsilon \, dx \\ &\leq C \int_{B(0,2)} |u^\varepsilon_{x_n} - 1| \left(1 + |D^2 u^\varepsilon D u^\varepsilon| + \varepsilon |D^2 u^\varepsilon|\right) \sigma^\varepsilon \, dx \\ &\leq C \left(\int_{B(0,2)} (u^\varepsilon_{x_n} - 1)^2 \sigma^\varepsilon \, dx \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{B(0,2) \cap \{|Du^\varepsilon|^2 \leq 1 - 2\delta\}} (u^\varepsilon_{x_n} - 1)^2 \sigma^\varepsilon \, dx \right)^{\frac{1}{2}} + C \left(\int_{B(0,2) \cap \{|Du^\varepsilon|^2 \geq 1 + 2\delta\}} (u^\varepsilon_{x_n} - 1)^2 \sigma^\varepsilon \, dx \right)^{\frac{1}{2}} \\ &\qquad + C \left(\int_{B(0,2) \cap \{1 + 2\delta \geq |Du^\varepsilon|^2 \geq 1 - 2\delta\}} (u^\varepsilon_{x_n} - 1)^2 \sigma^\varepsilon \, dx \right)^{\frac{1}{2}} \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

Owing to (4.7),

$$|A_1| \leq \frac{C}{\varepsilon^{\frac{1}{2}}} e^{\frac{-\mu\delta}{2\varepsilon}}.$$

Furthermore, (4.10) lets us estimate

$$|A_2| \le C \left(\int_{B(0,2)\cap\{|Du^\varepsilon|^2 \ge 1+2\delta\}} \sigma^\varepsilon \, dx \right)^{\frac12} \le \frac{C\lambda^{\frac12}}{\delta}.$$

Finally, on the set $\{1+2\delta \ge |Du^\varepsilon|^2 \ge 1-2\delta\}$ we have

$$(u^\varepsilon_{x_n} - 1)^2 \le C(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2 + C\delta^2. \tag{4.17}$$

Consequently,

$$|A_3| \le C \left(\int_{B(0,2)} (|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2 \sigma^\varepsilon \, dx + C\delta^2 \right)^{\frac12} \le C(\lambda^{\frac12} + \delta).$$

in view of the estimate (4.9).

Collecting the foregoing bounds, we conclude that

$$|A| \le \frac{C}{\varepsilon^{\frac12}} e^{\frac{-\mu\delta}{2\varepsilon}} + \frac{C\lambda^{\frac12}}{\delta} + C(\lambda^{\frac12} + \delta). \tag{4.18}$$

3. *Estimate of B.* To control this term, we first observe that

$$\begin{aligned} u^\varepsilon_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_n x_i} \zeta_{x_j} &= \tfrac12 \sum_{j=1}^{n-1} u^\varepsilon_{x_j} (|Du^\varepsilon|^2)_{x_n} \zeta_{x_j} + \tfrac12 u^\varepsilon_{x_n} (|Du^\varepsilon|^2)_{x_n} \zeta_{x_n} \\ &= \tfrac12 \sum_{j=1}^{n-1} u^\varepsilon_{x_j} (|Du^\varepsilon|^2)_{x_n} \zeta_{x_j} - \left(\tfrac12 \sum_{j=1}^{n-1} u^\varepsilon_{x_j} (|Du^\varepsilon|^2)_{x_j} + \varepsilon \Delta u^\varepsilon \right) \zeta_{x_n} \end{aligned}$$

according to the PDE (4.2). Consequently,

$$|u^\varepsilon_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_n x_i} \zeta_{x_j}| \le C |D' u^\varepsilon| |D^2 u^\varepsilon D u^\varepsilon| + C\varepsilon |D^2 u^\varepsilon|, \tag{4.19}$$

where $D'u^\varepsilon := (u^\varepsilon_{x_1}, \dots, u^\varepsilon_{x_{n-1}}, 0)$.

And so

$$\begin{aligned} |B| &\le \int_{B(0,2)} (|D'u^\varepsilon| |D^2 u^\varepsilon D u^\varepsilon| + C\varepsilon |D^2 u^\varepsilon|) \sigma^\varepsilon \, dx \\ &\le C \left(\int_{B(0,2)} |D'u^\varepsilon|^2 \sigma^\varepsilon \, dx \right)^{\frac12} + C\varepsilon^{\frac12} \\ &\le C \left(\int_{B(0,2)\cap\{|Du^\varepsilon|^2 \le 1-2\delta\}} |D'u^\varepsilon|^2 \sigma^\varepsilon \, dx \right)^{\frac12} + C \left(\int_{B(0,2)\cap\{|Du^\varepsilon|^2 \ge 1+2\delta\}} |D'u^\varepsilon|^2 \sigma^\varepsilon \, dx \right)^{\frac12} \\ &\qquad + C \left(\int_{B(0,2)\cap\{1+2\delta \ge |Du^\varepsilon|^2 \ge 1-2\delta\}} |D'u^\varepsilon|^4 \sigma^\varepsilon \, dx \right)^{\frac14} + C\varepsilon^{\frac12} \\ &=: B_1 + B_2 + B_3 + C\varepsilon^{\frac12}. \end{aligned}$$

As above,

$$|B_1| \le \frac{C}{\varepsilon^{\frac{1}{2}}} e^{\frac{-\mu\delta}{2\varepsilon}}, \ |B_2| \le \frac{C\lambda^{\frac{1}{2}}}{\delta}.$$

In addition, on the set $\{1+2\delta \ge |Du^\varepsilon|^2 \ge 1-2\delta\}$, we have

$$|D'u^\varepsilon|^2 \le 1 - (u_{x_n})^2 + 2\delta \le C|1-u^\varepsilon_{x_n}| + C\delta.$$

We therefore have from (4.17) that

$$|D'u^\varepsilon|^4 \le C(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2 + C\delta^2.$$

Consequently, estimate (4.9) lets us conclude that

$$|B_3| \le C(\lambda^{\frac{1}{4}} + \delta^{\frac{1}{2}}).$$

Combining all these estimates, we see that we conclude that

$$|B| \le \frac{C}{\varepsilon^{\frac{1}{2}}} e^{\frac{-\mu\delta}{2\varepsilon}} + \frac{C\lambda^{\frac{1}{2}}}{\delta} + C(\lambda^{\frac{1}{4}} + \delta^{\frac{1}{4}}) + C\varepsilon^{\frac{1}{2}}. \tag{4.20}$$

4 The last inequality and the similar bound (4.18) for the term A prove that $|u^\varepsilon_{x_n} - 1|$, and therefore $|u^\varepsilon_{x_n} - 1|^2$, are less than or equal to the right hand side of (4.13).

To estimate the other derivatives, we see from (4.12) that at the point x^0

$$|D'u^\varepsilon|^2 \le 1 - (u^\varepsilon_{x_n})^2 + \delta \le C|1-u^\varepsilon_{x_n}| + \delta.$$

This and the foregoing estimate for $|u^\varepsilon_{x_n} - 1|$ complete the proof of (4.13). □

5 Everywhere differentiability

5.1 Blow up limits. If $-\Delta_\infty u = 0$ in the viscosity sense in some open subset $U \subseteq \mathbb{R}^n$ and $B(x,r) \subset U$, we define

$$L^+_r(x) := \frac{\max_{\partial B(x,r)} u - u(x)}{r}, \ L^-_r(x) := \frac{u(x) - \min_{\partial B(x,r)} u}{r}$$

As proved for example in [C-E-G], the limits

$$L(x) := \lim_{r\to 0} L^+_r(x) = \lim_{r \to 0} L^-_r(x)$$

exist and are equal for each point $x \in U$. (We will see later that in fact $L(x) = |Du(x)|$).

The paper [C-E-G] proves the following theorem, asserting that any blow-up limit around any point $x \in U$ must be a linear function. See [C-E] for a simplifed proof.

THEOREM 5.1 *Let u be a viscosity solution of*

$$-\Delta_\infty u = 0 \quad \text{in } U$$

and select any point $x \in U$.

For each sequence $\{r_j\}_{j=1}^\infty$ converging to zero, there exists a subsequence $\{r_{j_k}\}_{k=1}^\infty$ such that

$$\frac{u(r_{j_k}y + x) - u(x)}{r_{j_k}} \to a \cdot y \qquad \text{locally uniformly,} \tag{5.1}$$

for some $a \in \mathbb{R}^n$ such that

$$|a| = L(x). \tag{5.2}$$

Since solutions of $-\Delta_\infty u = 0$ are locally Lipschitz continuous, the rescaled functions $u_r(y) := \frac{u(ry+x)-u(x)}{r}$ are locally bounded and Lipschitz continuous and consequently contain a locally uniformly convergent subsequence. Theorem 5.1 asserts that each such limit is linear, but does not prove that various blow-up limits, corresponding to different subsequences of radii going to zero, are the same (unless $L(x) = 0$).

5.2 Differentiability. This section resolves this uncertainty by proving the uniqueness of the blow-up limits (5.1).

LEMMA 5.2 *Assume $b \in \mathbb{R}^n$, $|b| = 1$. Let v be a smooth function satisfying*

$$\max_{B(0,1)} |v - b \cdot x| \leq \eta$$

for some small constant η. Then there exists a point $x^0 \in B(0,1)$ at which

$$|Dv(x^0) - b| \leq 6\eta$$

Proof. Define

$$w := b \cdot x - 3\eta|x|^2 + \alpha.$$

We select the constant α so that $v \geq w$ in $B(0,1)$, but $v(x^0) = w(x^0)$ at some interior point x^0. Then $Dv(x^0) = Dw(x^0) = b - 6\eta x^0$. □

THEOREM 5.3 *Let u be the unique viscosity solution of*

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } U \\ \quad\;\; u = g & \text{on } \partial U. \end{cases} \tag{5.3}$$

Then u is differentiable at each point in U.

Proof. 1. Select any point within U, which without loss we may assume is 0. Suppose that the blow up discussed in §5.1 does not produce a unique tangent plane at 0. This means there exist two sequences $\{r_j\}_{j=1}^{\infty}, \{s_k\}_{k=1}^{\infty}$, each converging to zero, for which

$$\max_{B(0,r_j)} \frac{1}{r_j}|u(x) - u(0) - a \cdot x| \to 0 \tag{5.4}$$

and

$$\max_{B(0,s_k)} \frac{1}{s_k}|u(x) - u(0) - b \cdot x| \to 0, \tag{5.5}$$

for distinct vectors $a, b \in \mathbb{R}^n$, with $|a| = |b| > 0$. We may assume without loss that

$$a = e_n, \ |b| = 1, \ b \neq e_n.$$

Define

$$\theta := |b - e_n| > 0. \tag{5.6}$$

2. Hereafter C denotes the constant on the right hand side of estimate (4.13). We now adjust various parameters to make the right hand side of this inequality small as compared with θ^2.

First select $\delta > 0$ so small that

$$C\delta^{\frac{1}{2}} \leq \frac{\theta^2}{24}. \tag{5.7}$$

Now fix $\lambda > 0$ so that

$$C\left(\frac{\lambda^{\frac{1}{2}}}{\delta} + \lambda^{\frac{1}{4}}\right) \leq \frac{\theta^2}{24}. \tag{5.8}$$

Next select $\varepsilon_1 > 0$ so that

$$C\left(\frac{e^{\frac{-\mu\delta}{2\varepsilon}}}{\varepsilon^{\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}\right) \leq \frac{\theta^2}{24}. \tag{5.9}$$

for all $0 < \varepsilon \leq \varepsilon_1$.

We then use (5.4) (with $a = e_n$) to select a radius $r_j > 0$ for which

$$\max_{B(0,r_j)} \frac{1}{r_j}|u(x) - u(0) - x_n| \leq \frac{\lambda}{2}.$$

We may without loss assume that $r_j = 2$ and that $u(0) = 0$, as we can otherwise rescale and consider the function $\frac{u(r_j x) - u(0)}{r_j}$. Hence

$$\max_{B(0,2)} |u - x_n| \leq \frac{\lambda}{2}. \tag{5.10}$$

Now pick $\varepsilon_2 > 0$ so that

$$\max_{B(0,2)} |u^\varepsilon - x_n| \leq \lambda. \tag{5.11}$$

for all $0 < \varepsilon \leq \varepsilon_2$.

3. We introduce yet another constant $\eta > 0$, picked so that

$$12\eta + 36\eta^2 \leq \delta, \; 6\eta \leq \delta, \tag{5.12}$$

and

$$72\eta^2 \leq \frac{\theta^2}{4}. \tag{5.13}$$

In view of (5.5), we can find a (possibly very small) radius $0 < s < 1$ for which

$$\max_{B(0,s)} \frac{1}{s} |u - b \cdot x| \leq \frac{\eta}{2}.$$

We select $\varepsilon_3 > 0$ so that

$$\max_{B(0,s)} \frac{1}{s} |u^\varepsilon - b \cdot x| \leq \eta, \tag{5.14}$$

for all $0 < \varepsilon \leq \varepsilon_3$, and hereafter take

$$\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}. \tag{5.15}$$

Rescaling (5.14) to the unit ball and applying the Lemma, we secure a point $x^0 \in B(0,s) \subseteq B(0,1)$ at which

$$|Du^\varepsilon(x^0) - b| \leq 6\eta. \tag{5.16}$$

Then since $|b| = 1$, we have

$$|Du^\varepsilon(x^0)|^2 \leq (1 + 6\eta)^2 \leq 1 + \delta \tag{5.17}$$

according to (5.12). Furthermore, $|Du^\varepsilon(x^0)| \geq 1 - 6\eta$ and therefore

$$|Du^\varepsilon(x^0)|^2 \geq 1 - \delta, \tag{5.18}$$

again owing to (5.12).

4. Now (5.17) and (5.18) allow us to invoke the key estimate (4.13):

$$|Du^\varepsilon(x^0) - e_n|^2 \leq C \left(\frac{e^{\frac{-\mu\delta}{2\varepsilon}}}{\varepsilon^{\frac{1}{2}}} + \frac{\lambda^{\frac{1}{2}}}{\delta} + \lambda^{\frac{1}{4}} + \delta^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \right).$$

In view of our choices (5.7), (5.8) and (5.9), it follows that

$$|Du^\varepsilon(x^0) - e_n|^2 \leq \frac{\theta^2}{8}. \tag{5.19}$$

Using (5.6), (5.13), (5.16) and (5.19), we at last reach the contradiction that

$$\theta^2 = |b - e_n|^2 \leq 2|Du^\varepsilon(x^0) - b|^2 + 2|Du^\varepsilon(x^0) - e_n|^2 \leq 72\eta^2 + \frac{\theta^2}{4} \leq \frac{\theta^2}{2}.$$

□

Our paper [E-S] presents a simpler proof of the everywhere differentiability.

6 The infinity Laplacian PDE as a parabolic equation

This section provides heuristics to justify our claim that the infinity Laplacian PDE

$$-\Delta_\infty u = 0 \tag{6.1}$$

should be regarded as a *parabolic*, and not an elliptic, equation. (G. Aronsson has made a similar observation in his old paper [A], although for different reasons.)

6.1 Linearization. The only reasonable way to assert that a given nonlinear PDE is elliptic or parabolic or hyperbolic at a particular solution is to classify, if possible, the type of its linearization around this solution. We therefore consider the formal linearization of (6.1), which is the PDE

$$Lv := -u_{x_i}u_{x_j}v_{x_ix_j} - 2u_{x_i}u_{x_ix_j}v_{x_j} = 0 \tag{6.2}$$

for the "variation" v.

We contend that L is a parabolic equation, at least generically. Indeed the the second-order term $u_{x_i}u_{x_j}v_{x_ix_j}$ corresponds to diffusion along the line parallel to the gradient Du, whereas the first-order term $-2u_{x_i}u_{x_ix_j}v_{x_j}$ corresponds to transport in the direction $-D^2uDu$. According to the infinity Laplacian equation (6.1), the direction of diffusion $\pm Du$ is orthogonal to the direction of transport.

The linearized PDE (6.2) is therefore analogous to the one-dimensional linear heat equation

$$v_t = v_{xx},$$

except that (6.2) has variable coefficients, depending upon u, and holds in many variables. We may think of the direction of $-D^2uDu$ as the "time-like" direction and the perpendicular directions, including that of Du, as "space-like". In particular a critical point x^0, where $|Du(x^0)| = 0$, is at "time-like infinity". Several of our rigorous assertions are consistent with this interpretation, most notably the exponential estimate (3.9) which asserts that the value of ρ^ε is negligible at points $y \in \partial U$ where $|Du^\varepsilon(y)| < |Du^\varepsilon(x^0)|$. Such points are "forwards in time" for x^0 and so should not affect the solution at that point.

(If our smooth solution u of (6.1) happens also to be a solution of the eikonal equation $|Du| \equiv \alpha$ for some constant α, the time-like term does not appear and the linearization is a degenerate elliptic equation.)

6.2 Finite difference approximation. Our revisiting a standard finite difference approximation for the infinity Laplacian also reveals the parabolic structure.

Fix a step size $h > 0$ and define the nonlinear finite difference operator operator

$$\Delta_\infty^h u(x) := \frac{1}{h^2}\big(\max_{B(x,h)} u + \min_{B(x,h)} u - 2u(x)\big). \tag{6.3}$$

Then

$$\Delta_\infty^h u(x) = \frac{1}{h^2}(u(x^+) + u(x^-) - 2u(x)), \tag{6.4}$$

the points $x^\pm$ are selected so that

$$u(x^+) = \max_{B(x,h)} u,\ u(x^-) = \min_{B(x,h)} u.$$

If u is smooth and $Du \neq 0$, we have

$$\lim_{h\to 0} \Delta_\infty^h u = \frac{\Delta_\infty u}{|Du|^2} : \tag{6.5}$$

see for instance Armstrong-Smart [A-S].

LEMMA 6.1 *If u is smooth and $Du(x) \neq 0$, then*

$$x^+ = x + h\frac{Du}{|Du|} + h^2\left(\frac{D^2uDu}{|Du|^2} - \frac{\Delta_\infty uDu}{|Du|^4}\right) + O(h^3) \tag{6.6}$$

and

$$x^- = x - h\frac{Du}{|Du|} + h^2\left(\frac{D^2uDu}{|Du|^2} - \frac{\Delta_\infty uDu}{|Du|^4}\right) + O(h^3), \tag{6.7}$$

Du and D^2u evaluated at the point x.

Proof. Without loss, we may assume $x = 0$. Then

$$\begin{aligned} x^+ &= h\frac{Du(x^+)}{|Du(x^+)|} \\ &= h\left(Du(0) + D^2u(0)x^+ + O(h^2)\right)\left(\frac{1}{|Du(0)|} - \frac{Du(0)\cdot D^2u(0)x^+}{|Du(0)|^3} + O(h^2)\right), \end{aligned}$$

and so

$$x^+ = h\frac{Du(0)}{|Du(0)|} + O(h^2).$$

Plugging this into the previous expansion, we deduce (6.6). The derivation of (6.7) is similar. □

We observe that in view of (6.6) and (6.7) the difference scheme (6.4) is that for a parabolic PDE, involving $O(h)$ steps in the "space-like" directions $\pm Du$ and an $O(h^2)$ step in the "time-like" direction $-D^2uDu$. It is straightforward to check the consistency condition that (6.5) follows from (6.6), (6.7).

6.3 Stochastic differential equations. We introduce next a stochastic differential equation, which provides a probabilistic interpretation of ρ^ε and σ^ε:

$$\begin{cases} d\mathbf{X}^\varepsilon = Du^\varepsilon(\mathbf{X}^\varepsilon)dW + D^2u^\varepsilon(\mathbf{X}^\varepsilon)Du^\varepsilon(\mathbf{X}^\varepsilon)dt + (2\varepsilon)^{\frac{1}{2}}d\mathbf{W} \\ \mathbf{X}^\varepsilon(0) = x^0, \end{cases} \tag{6.8}$$

where W is a one-dimensional Brownian motion and $\mathbf{W} = (W^1, \ldots, W^n)$ is an independent n-dimensional Brownian motion.

Then ρ^ε is the density of the distribution of $\mathbf{X}^\varepsilon(\tau)$, where $\tau = \tau^\varepsilon_{x^0}$ is the first hitting time for ∂V. Furthermore if $E \subset V$ is a Borel set, then

$$\int_E \sigma^\varepsilon\, dx = E\left(\int_0^\tau \chi_{\{\mathbf{X}^\varepsilon(t)\in E\}}dt\right)$$

records the amount of time that the process $\mathbf{X}^\varepsilon$ spends within E before exiting V.

We can check using Ito's calculus that $Du^\varepsilon(\mathbf{X}^\varepsilon)$ is a martingale, although in general $u^\varepsilon(\mathbf{X}^\varepsilon)$ is not. This is why the formula (3.4) for the gradient $Du^\varepsilon(x^0)$ is exact, whereas the expression (3.14) for $u^\varepsilon(x^0)$ has an error term (which is however small as $\varepsilon \to 0$).

7 Some numerical experiments

In a series of experiments we have studied numerically the limiting behavior of σ^ε and ρ^ε as $\varepsilon \to 0$. We employed both a monotone and a second-order finite difference scheme, and only report computations for which both methods gave nearly identical results.

7.1 A monotone scheme. A. Oberman's monotone finite difference scheme [O] for the normalized infinity Laplacian PDE is easily adapted to our case: we need only multiply his finite difference operator by a suitable approximation of $|Du|^2$. Given a step size $h > 0$, an integer $d > 0$ and a function $u : h\mathbb{Z}^2 \to \mathbb{R}$, we therefore define

$$\Delta_\infty^{h,d} u(x) := \frac{1}{12(hd)^4}\left(\max_{z\in N(x)}(u(z)-u(x))^3 + \min_{z\in N(x)}(u(z)-u(x))^3\right),$$

where

$$N(x) := \{z \in h\mathbb{Z}^2 \mid h(d-1/2) \le |x-z| \le h(d+1/2)\}.$$

It is easy to see that $\Delta_\infty^{h,d}$ is monotone. Furthermore, for any smooth function φ, we have $\Delta_\infty^{h,d}\varphi \to \Delta_\infty\varphi$ locally uniformly as $d \to \infty$ and $hd \to 0$. Combining $\Delta_\infty^{h,d}$ with the standard 5-point Laplacian Δ^h, we obtain a monotone finite difference scheme of the form

$$-\Delta_\infty^{h,d} u^{h,d} - \varepsilon \Delta^h u^{h,d} = 0.$$

A theorem of Barles and Souganidis [B-S] immediately implies the convergence of this scheme.

7.2 A second-order scheme. To obtain a higher-order scheme, we exploit the variational structure of the regularized PDE (1.2). If we multiply by the integration factor $\exp(\frac{1}{2\varepsilon}|Du^\varepsilon|^2)$, we obtain

$$-\operatorname{div}\left(e^{\frac{1}{2\varepsilon}|Du^\varepsilon|^2} Du^\varepsilon\right) = 0; \tag{7.1}$$

and this is the Euler-Lagrange equation for minimizers of the functional

$$\int_U e^{\frac{1}{2\varepsilon}|Dv|^2}|Dv|^2\,dx. \tag{7.2}$$

We can now construct a second-order convergent finite difference approximation for (7.1) using standard techniques (see for example Hackbusch [H] or LeVeque [L]). We in particular selected a second-order accurate discretization of (7.1) and then solved the Euler-Lagrange equations for the discrete variational problem.

We must however be very careful when implementing such a scheme, as the fast growth of $\exp\left(\frac{1}{2\varepsilon}|Dv|^2\right)$ increases the condition number of the linearization. Preconditioning is required to obtain an accurate solution when ε is small. Even with this adjustment, numerical instability manifests itself as a failure of the maximum principle for the adjoint of the linearization when the step size h is insufficiently small relative to ε.

7.3 Experimental results. For each trial we took several small values of ε and approximated u^ε, σ^ε and ρ^ε for fixed boundary data on the square

$$Q := \{x \in \mathbb{R}^2 \mid |x_1|, |x_2| < 1\}.$$

Computation 1. In our first experiment, we set $x^0 = 0$ and used boundary data given by the argument function

$$u(x) := \arctan\left(\frac{x_2}{2 + x_1}\right).$$

Since u solves the regularized PDE (1.2) in $\mathbb{R}^2 - \{(-2, 0)\}$ for all $\varepsilon > 0$, we expect that σ^ε converges as $\varepsilon \to 0$ to the solution σ of

$$\begin{cases} -(u_{x_i} u_{x_j} \sigma)_{x_i x_j} - 2(u_{x_i} u_{x_i x_j} \sigma)_{x_j} = \delta_{x^0} & \text{in } Q, \\ \sigma = 0 & \text{on } \partial Q. \end{cases} \tag{7.3}$$

This is exactly what appears to happen in Figure 1 below.

Computation 2. As a second numerical experiment, we put $x^0 = (1/10, 1/2)$ and used boundary data given by

$$u(x) := x_1^{4/3} - x_2^{4/3}, \tag{7.4}$$

an infinity harmonic function discovered by Aronsson that is nonsmooth along the coordinate axes $\{x_1 x_2 = 0\}$ (which we regard as "weak shocks").

We argue heuristically for this example that σ^ε and ρ^ε csnnot concentrate solely within the first quadrant $Q \cap \{x_1, x_2 > 0\}$ as $\varepsilon \to 0$, and therefore trajectories of the stochastic differential equation (6.8) with positive probability diffuse across the forming weak shocks along the coordinate axes. To see this, remember from (3.4) that

$$Du^\varepsilon(x^0) = \int_{\partial Q} Du^\varepsilon \rho^\varepsilon \, dS. \tag{7.5}$$

We assume now that Du^ε is close to Du along ∂Q. Since then $u^\varepsilon_{x_1} > 0 > u^\varepsilon_{x_2}$ and since $|Du^\varepsilon| > |Du^\varepsilon(x^0)|$ on $\partial Q \cap \{x_1, x_2 > 0\}$, the identity (7.5) could not be true as $\varepsilon \to 0$ if ρ^ε were to concentrate only on $\partial Q \cap \{x_1, x_2 > 0\}$.

Observe also that if we set $\varepsilon = 0$ in the stochastic differential equation (6.8), the transport vector

$$D^2uDu = \frac{4}{27}(x_1^{-1}, x_2^{-1}),$$

not integrable near the coordinate axes $\{x_1x_2 = 0\}$; whereas the diffusion matrix

$$Du \otimes Du = \frac{16}{9}\begin{pmatrix} x_1^{2/3} & 0 \\ 0 & x_2^{2/3} \end{pmatrix},$$

is bounded. So presumably a competition occurs as $\varepsilon \to 0$ between the decay of the diffusion and the growth of the transport in (6.8); and in the limit some positive portion of the mass of σ^ε must remain outside of the first quadrant.

It appears from the numerical data that σ^ε converges as $\varepsilon \to 0$ to a function σ that solves (7.3) in $Q - \{x_1x_2 = 0\}$, but is singular on $\{|Du| > |Du(x^0)|\} \cap \{x_1x_2 = 0\}$. There are corresponding singularities in the limit of the ρ^ε at the four points where these "weak shocks" hit the boundary. This is most apparent in the bottom image in Figure 2, in which we see cusps forming in the graph the of ρ^ε as $\varepsilon \to 0$.

Computation 3. In our final experiment, we set $x^0 = (0, 1/10)$ and used the boundary data

$$u(x) := \frac{(1 + rx_1)^{4/3} - (rx_2)^{4/3} - 1}{r},$$

for small $r > 0$. That is, we zoomed in to a small neighborhood of a point on the weak shocks of the Aronsson function (7.4) and get a closer view of the apparent singularities in σ^ε and ρ^ε forming as $\varepsilon \to 0$. See Figure 3.

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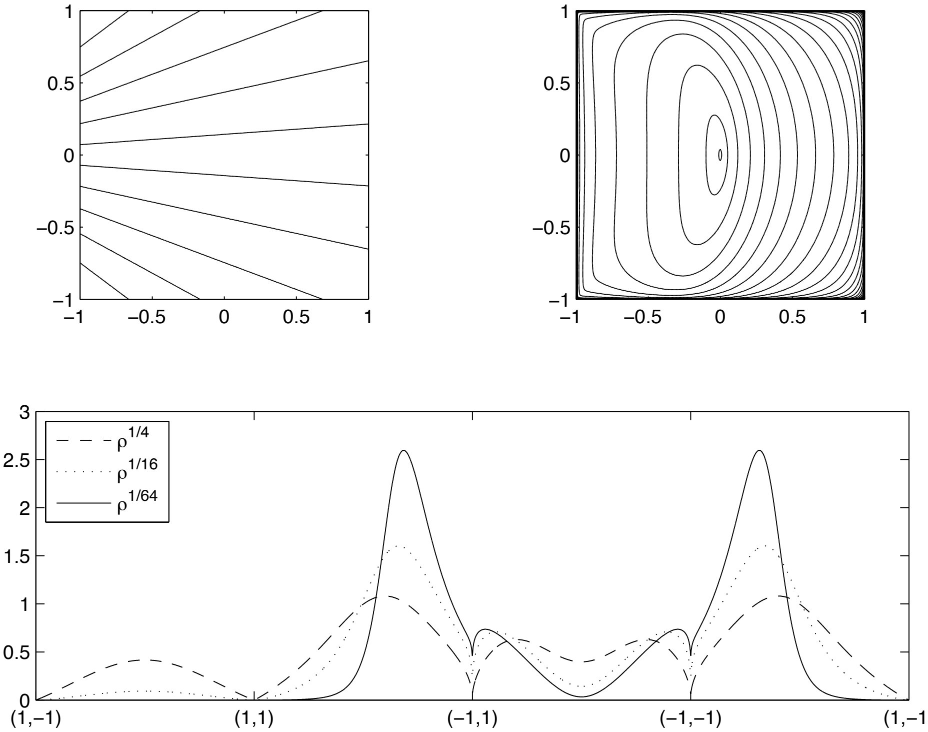

Figure 1: Clockwise from the upper left are the level sets of u^ε for $\varepsilon = 1/64$, the level sets of σ^ε for $\varepsilon = 1/64$ and ρ^ε for $\varepsilon = 1/4, 1/16, 1/64$. The boundary data are $g(x, y) := \arctan(y/(x + 2))$ and the initial point is $x^0 = 0$.

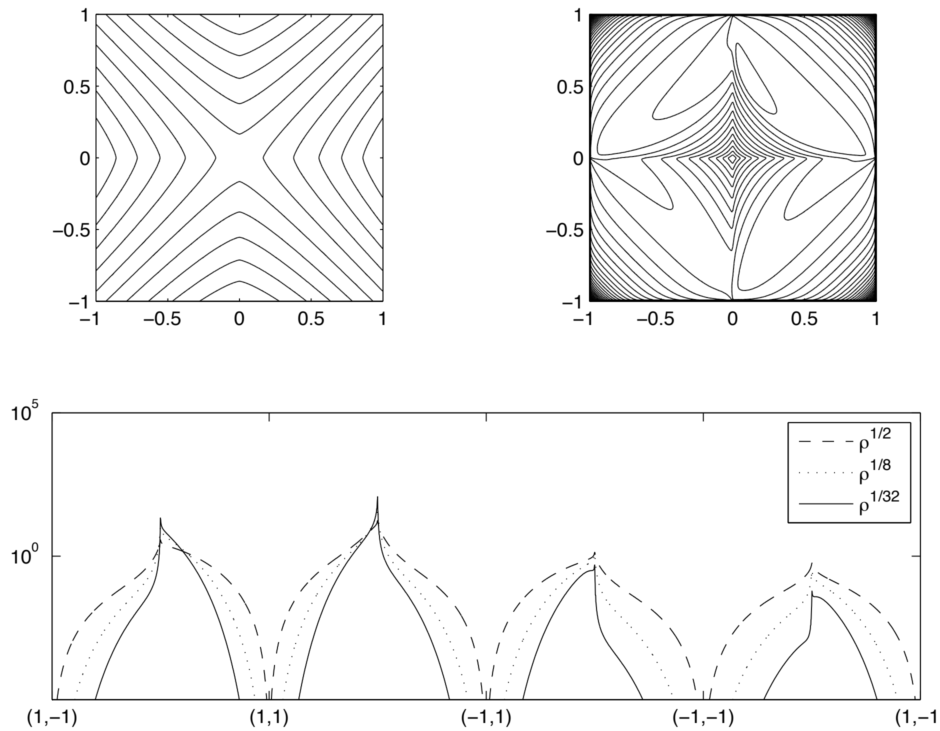

Figure 2: Clockwise from the upper left are the level sets of u^ε for $\varepsilon = 1/32$, the level sets of σ^ε for $\varepsilon = 1/32$, and ρ^ε for $\varepsilon = 1/2, 1/8, 1/32$. The boundary data are given by $g(x, y) := x^{4/3} - y^{4/3}$ and the initial point is $x^0 = (1/10, 1/2)$.

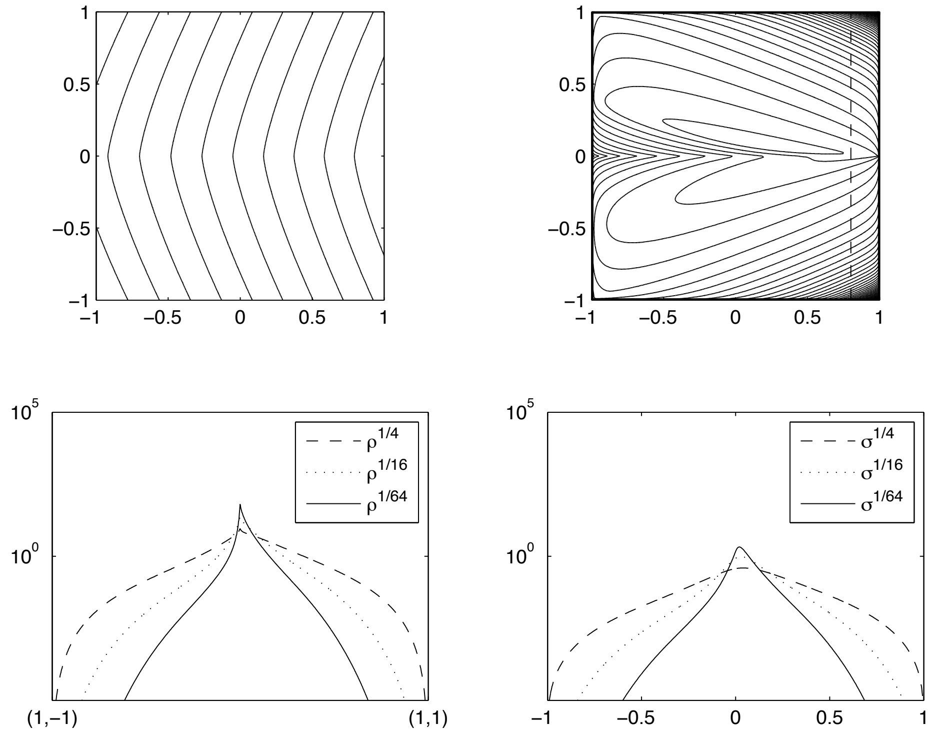

Figure 3: Clockwise from the upper left are the level sets of u^ε for $\varepsilon = 1/64$, the level sets of σ^ε for $\varepsilon = 1/64$, the cross sections of σ^ε on $\{x_1 = 8/10\}$ (indicated by the dotted line), and ρ^ε along the right-most edge of the domain. The boundary data are $g(x, y) := r^{-1}[(1 + rx)^{4/3} - (ry)^{4/3}]$ for $r = 1/10$ and the initial point is $x^0 = (0, 1/10)$.