NOTES FOR MATH 6140 VISCOSITY SOLUTIONS

CHARLES K SMART

These are (still rough) lecture notes describing some of the material covered in Math 6140 Viscosity Solutions in Spring 2015 at Cornell.

Contents

1. The distance function

We begin by studying a trivial control problem. Suppose we would like to move a particle $x \in U$ to the complement $\mathbb{R}^d \setminus U$ of a bounded open set $U \subseteq \mathbb{R}^d$ as quickly as possible subject to the velocity constraint $|\dot{x}| \leq 1$. Of course, the optimal escape time as a function of x is the distance function

$$w(x) = \min_{y \in \mathbb{R}^d \setminus U} |x - y|.$$

We claim that w is the unique solution of the partial differential equation

$$\begin{cases} |Dw| = 1 & \text{in } U \\ w = 0 & \text{on } \partial U. \end{cases}$$

This is suspicious, since w is not differentiable!

We first give a local characterization of w. Observe that any trajectory from x to $\mathbb{R}^d \setminus U$ can be divided into two pieces by cutting at the first crossing of $\partial B(x, r)$, provided $r > 0$ is small enough that $\bar{B}(x, r) \subseteq U$. This gives a lemma.

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Lemma 1.1. *The distance function w is Lipschitz continuous and satisfies*

$$w(x) = r + \min_{\partial B(x,r)} w$$

whenever $0 < r < w(x)$.

Proof. The triangle inequality gives

$$w(x) = w(y) + |y - x|$$

for all $x, y \in \mathbb{R}^d$. Thus w is Lipschitz continuous and satisfies

$$u(x) \le r + \min_{\partial B(x,r)} u$$

for $x \in \mathbb{R}^d$ and $r > 0$. If $0 < r < u(x)$, then we can choose $z \in \mathbb{R}^d \setminus U$ such that $u(x) = |x - z|$. Letting

$$y = x + r \frac{z - x}{|z - x|},$$

we compute $u(x) = |x - z| = |x - y| + |y - z| \ge r + u(y) \ge r + \min_{\partial B(x,r)} u$. □

This lemma is sometimes called a dynamic programming principle. It says that the globally optimal escape time should be locally optimal. It turns out that local optimality implies global optimality.

Theorem 1.2. *The distance function w is the unique continuous function satisfying*

$$\begin{cases} \lim_{r \to 0} \dfrac{w(x) - \min_{\partial B(x,r)} w}{r} = 1 & \text{if } x \in U \\ w(x) = 0 & \text{if } x \in \mathbb{R}^d \setminus U. \end{cases} \tag{1.1}$$

Proof. Suppose for contradiction that w and v are distinct continuous functions satisfying (1.1). By symmetry, we may assume that $\sup(w - v) > 0$. Since w and v are continuous and vanish outside the bounded open set U, we may select $\tau \in (0, 1)$ and $x \in U$ such that

$$\max(\tau w - v) = (\tau w - v)(x) > 0.$$

Let $y_r \in \partial B(x, r)$ satisfy

$$\min_{\partial B(x,r)} w = w(y_r) \quad \text{and} \quad \min_{\partial B(x,r)} v \le v(y_r).$$

Using (1.1), we obtain

$$w(x) - w(y_r) = r + o(r)$$

and

$$v(x) - v(y_r) \ge r + o(r).$$

Using $\tau w(x) - v(x) \ge \tau w(y_r) - v(y_r)$, we conclude

$$0 \le (\tau - 1)r + o(r).$$

Since $\tau - 1 < 0$, this is impossible. □

Our control-theoretic local characterization of the distance function approximates a partial differential equation in the following sense.

Lemma 1.3. *If $v \in C(\mathbb{R}^d)$ is differentiable at x, then*

$$\lim_{r \to 0} \frac{v(x) - \min_{\partial B(x,r)} v}{r} = |Dv(x)|.$$

Proof. From the Taylor expansion

$$v(y) = v(x) + Dv(x) \cdot (y - x) + o(|y - x|),$$

we compute

$$\min_{\partial B(x,r)} v = v(x) - |Dv(x)|r + o(r). \qquad \square$$

In particular, we see that the distance function w satisfies $|Dw| = 1$ wherever it is differentiable. This happens quite often.

Theorem 1.4 (Rademacher)**.** *Lipschitz continuous functions on $\mathbb{R}^d$ are differentiable Lebesgue almost everywhere.*

It is difficult to interpret $|Dw| = 1$ correctly.

Example 1.5. The boundary value problem

$$\begin{cases} |w'| = 1 & \text{in } (-1,1) \\ w = 0 & \text{on } \mathbb{R} \setminus (-1,1) \end{cases}$$

has no solution in $C(\mathbb{R}) \cap C^1((-1,1))$. If we relax the first condition to holding almost everywhere, there are infinitely many solutions $C^{0,1}(\mathbb{R})$.

Nonetheless, the distance function is still a natural solution.

Example 1.6. For $\varepsilon > 0$, the boundary value problem

$$\begin{cases} |w_\varepsilon'| = 1 + \varepsilon w_\varepsilon'' & \text{in } (-1,1) \\ w_\varepsilon = 0 & \text{on } \mathbb{R} \setminus (-1,1) \end{cases}$$

has unique solution in $C(\mathbb{R}) \cap C^1((-1,1))$ given by

$$w_\varepsilon(x) = \begin{cases} 1 - |x| + \varepsilon(e^{-1/\varepsilon} - e^{-|x|/\varepsilon}) & \text{if } x \in (-1,1) \\ 0 & \text{if } x \in \mathbb{R} \setminus (-1,1). \end{cases}$$

Note that w_ε converges uniformly as $\varepsilon \to 0$ to $w(x) = \max\{0, 1 - |x|\}$.

2. Differential inequalities

Consider the first order partial differential inequality

$$F(x, u, Du) \le 0 \quad \text{in } U,$$

where $U \subseteq \mathbb{R}^d$ is open and $F \in C(U \times \mathbb{R} \times \mathbb{R}^d)$ is an arbitary nonlinearity. We wish to allow merely semicontinuous functions to satisfy this inequality.

Definition 2.1. A function $u : U \to \mathbb{R}$ is upper semicontinuous if and only if

$$u(x) = \lim_{r \to 0} \sup_{B(x,r)} u \quad \text{for } x \in U.$$

Let $C^+(U)$ denote the set of all upper semicontinuous functions on U.

Definition 2.2. A function $u \in C^+(U)$ satisfies

$$F(x, u, Du) \le 0 \quad \text{in } U$$

in the sense of viscosity if and only if

$$F(x, \varphi(x), D\varphi(x)) \le 0$$

holds whenever $\varphi \in C^1(U)$, $x \in U$, and $\max_U(u - \varphi) = (u - \varphi)(x) = 0$.

Remark 2.3. When $\max_U(u-\varphi) = (u-\varphi)(x) = 0$, we say that φ touches u from above at x in U. We obtain an equivalent definition if we replace maximum with local maximum.

Theorem 2.4. *If $u \in C^1(U)$, then the classical and viscosity interpretations of $F(x,u,Du) \leq 0$ in U are equivalent.*

Proof. Suppose u satisfies $F(x,u,Du) \leq 0$ in U in the sense of viscosity. Since u touches itself from above at every $x \in U$, the definition implies

$$F(x,u(x),Du(x)) \leq 0 \quad \text{for } x \in U.$$

On the other hand, suppose u satisfies $F(x,u,Du) \leq 0$ in U in the classical sense and $\varphi \in C^1(U)$ touches u from above at $x \in U$. Since $\max(u-\varphi) = (u-\varphi)(x) = 0$, we must have $u(x) = \varphi(x)$ and $Du(x) = D\varphi(x)$. Thus $F(x,\varphi(x),D\varphi(x)) = F(x,u(x),Du(x)) \leq 0$. □

Remark 2.5. The theorem implies that the viscosity interpretation is a relaxation of the classical interpretation. In light of this, we omit the phrase "in the sense of viscosity" in the sequel.

Example 2.6. Both of $u(x) = \pm|x|$ satisfy $|Du| - 1 \leq 0$ in $\mathbb{R}^d$.

We also wish to consider the opposite inequality

$$F(x,u,Du) \geq 0 \quad \text{in } U.$$

A suitable definition of viscosity solution can be obtained by reversing the inequalities above, and in the process defining the lower semicontinuous functions $C^-(U)$ and touching from below. Alternatively, we can use make use of symmetry.

Definition 2.7. Interpret $F(x,u,Du) \geq 0$ as $\bar{F}(x,v,Dv) \leq 0$, where $v = -u$ and $\bar{F}(x,s,p) = -F(x,-s,-p)$.

Example 2.8. Only one of $u(x) = \pm|x|$ satisfies $|Du| - 1 \geq 0$ in $\mathbb{R}^d$.

Definition 2.9. Interpret $F(x,u,Du) = 0$ as the conjunction of $F(x,u,Du) \leq 0$ and $F(x,u,Du) \geq 0$.

3. Strict comparison

Observe that, if $u, v \in C^1(U)$ satisfy $F(x,u,Du) < F(x,v,Dv)$ in U, then v does not touch u in U. This is a consequence of the proof of Theorem 2.4. This is also true when u and v are merely semicontinuous, provided the nonlinearity F is sufficiently continuous and the inequality is uniform.

Theorem 3.1. *Suppose $U \subseteq \mathbb{R}^d$ is open and $F \in C(U \times \mathbb{R} \times \mathbb{R}^d)$ satisfies*

$$|F(x,s,p) - F(y,t,q)| \leq \omega(|x-y|(1+|s|+|p|) + |s-t| + |p-q|)$$

for some modulus of continuity ω. If $u \in C^+(U)$, $v \in C^-(U)$, and $\delta > 0$ satisfy

$$F(x,u,Du) \leq 0 \quad \text{and} \quad F(x,v,Dv) - \delta \geq 0 \quad \text{in } U,$$

then v does not touch u from above at any point in U.

We need a perturbation lemma.

Lemma 3.2. *If $u \in C^+(U)$ satisfies $F(x,u,Du) \leq 0$ in U and $f \in C^1(U)$, then $v = u + f$ satisfies $G(x,v,Dv) \leq 0$ in U, where $G(x,s,p) = F(x, s - f(x), p - Df(x))$.*

Proof. If $\varphi \in C^1(U)$ touches $u+f$ from above at x, then $\psi = \varphi - f$ touches u from above at x. Thus

$$\begin{aligned} 0 &\geq F(x, \psi(x), D\psi(x)) \\ &= F(x, \varphi(x) - f(x), D\varphi(x) - Df(x)) \\ &= G(x, \varphi(x), D\varphi(x)), \end{aligned}$$

as required. □

Proof of Theorem 3.1. Suppose for contradiction that v touches u from above in U. We may assume that $U = B(0,1)$ and the touching occurs at 0.

Step 1. By Lemma 3.2 and the continuity of F, the perturbation

$$w(x) = v(x) + \frac{|x|^2}{2}$$

satisfies

$$F(x, w, Dw) - \frac{\delta}{2} \geq 0 \quad \text{in } B(0,r)$$

for some $r > 0$.

Step 2. For all small $\varepsilon > 0$, the function

$$\Phi_\varepsilon(x,y) = u(x) - w(y) - \frac{|x-y|^2}{2\varepsilon}$$

attains its maximum on $B(0,r)^2$ at some point $(x_\varepsilon, y_\varepsilon) \in B(0,r)^2$ satisfying

$$|u(x_\varepsilon)| \leq O(1), \quad |u(x_\varepsilon) - v(y_\varepsilon)| \leq o(1), \quad \text{and} \quad |x_\varepsilon - y_\varepsilon| \leq o(\varepsilon^{1/2})$$

as $\varepsilon \to 0$.

Step 3. The smooth function

$$x \mapsto w(y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon} + \Phi_\varepsilon(x_\varepsilon, y_\varepsilon)$$

touches u from above at x_ε. Using the viscosity definition, we conclude that

$$F\left(x_\varepsilon, u(x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right) \leq 0.$$

Similarly, the smooth function

$$y \mapsto u(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon} - \Phi_\varepsilon(x_\varepsilon, y_\varepsilon)$$

touches w from below at y_ε and we conclude that

$$F\left(y_\varepsilon, w(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right) - \frac{\delta}{2} \geq 0.$$

Combining these inequalities and using the continuity of F, we estimate

$$\frac{\delta}{2} \leq \omega\left(|x_\varepsilon - y_\varepsilon|\left(1 + |u(x_\varepsilon)| + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\right) + |u(x_\varepsilon) - w(y_\varepsilon)|\right).$$

Using step 2, we conclude that

$$\frac{\delta}{2} \leq o(1)$$

as $\varepsilon \to 0$, which is impossible. □

4. The Eikonal equation

We now show, for $U \subseteq \mathbb{R}^d$ open and bounded, that the distance function

$$w(x) = \min_{y \in \mathbb{R}^d \setminus U} |x - y|$$

is the unique solution of

$$\begin{cases} |Dw| - 1 = 0 & \text{in } U \\ w = 0 & \text{on } \mathbb{R}^d \setminus U. \end{cases} \tag{4.1}$$

We first check that w is a solution.

Lemma 4.1. *If $\varphi \in C^1(U)$ and $w - \varphi$ has a local maximum (minimum) at $x \in U$, then $|D\varphi(x)| - 1 \leq (\geq) 0$.*

Proof. Using the differentiablity of φ at x, we see that

$$\min_{\partial B(x,r)} \varphi = \varphi(x) - r|D\varphi(x)| + o(r)$$

for all $r > 0$. In the event of a local maximum, (1.1) implies

$$1 = \frac{u(x) - \min_{\partial B(x,r)} u}{r} \geq \frac{\varphi(x) - \min_{\partial B(x,r)} \varphi}{r} = |D\varphi(x)| + o(1)$$

as $r \to 0$. In the event of the local minimum, the inequality is reversed. □

We next check that w is unique by proving a comparison principle.

Lemma 4.2. *If $u \in C^+(U)$ and $v \in C^-(U)$ satisfy*

$$\begin{cases} |Du| - 1 \leq 0 & \text{in } U \\ |Dv| - 1 \geq 0 & \text{in } U \\ u \leq v & \text{on } \partial U, \end{cases}$$

then $u \leq v$ in U.

Proof. Step 1. We first claim that $v \geq 0$. Otherwise, lower semicontinuity would imply $\min_U v = v(x) < 0$ for some $x \in U$. Thus the constant function $\varphi = v(x)$ would touch v from below x in U and satisfy $|D\varphi| - 1 < 0$, contradicting the definition of $|Dv| - 1 \geq 0$.

Step 2. For $\delta \in (0, 1)$, let $u_\delta = (1 - \delta)u$. Using $v \geq 0$, we see that

$$\begin{cases} |Du_\delta| - 1 + \delta \leq 0 & \text{in } U \\ u_\delta \leq v & \text{on } \partial U. \end{cases}$$

Thus $u_\delta \leq v$ follows from Theorem 3.1. Sending $\delta \to 0$ gives $u \leq v$. □

Combining the above lemmas, we obtain the following theorem.

Theorem 4.3. *The distance function is the unique solution of (4.1).*

5. Comparison

We have already seen that comparison holds for the Eikonal equation in bounded domains. We now give two general examples of comparison useful in game theory applications. Assume that $H \in C(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies

$$|H(x,p) - H(y,q)| \leq \omega(|x-y||p| + |p-q|),$$

for some modulus of continuity ω.

Theorem 5.1. *If $U \subseteq \mathbb{R}^d$ is open, $u \in C^+(\bar{U})$ and $v \in C^-(\bar{U})$ are bounded, and*

$$\begin{cases} u - H(x, Du) \leq 0 & \text{in } U \\ v - H(x, Dv) \geq 0 & \text{in } U \\ u \leq v & \text{on } \partial U, \end{cases}$$

then $u \leq v$ in U.

Proof. Suppose for contradiction that $\sup_U (u - v) \geq \delta > 0$.

Step 1. Using Lemma 3.2, the perturbation

$$v_\varepsilon(x) = v(x) + \varepsilon\sqrt{1 + |x|^2}$$

satisfies

$$v_\varepsilon - H(x, Dv_\varepsilon) - \omega(\varepsilon) \geq 0 \quad \text{in } U.$$

Step 2. Since $u - v$ is bounded, we see that

$$\sup_U (u - v_\varepsilon) = \max_U (u - v_\varepsilon) \to \delta,$$

as $\varepsilon \to 0$.

Step 3. For small $\varepsilon > 0$, the vertical translation

$$w_\varepsilon = v_\varepsilon + \max_U (u - v_\varepsilon)$$

satisfies

$$w_\varepsilon - H(x, Dw_\varepsilon) - \frac{\delta}{2} \geq 0 \quad \text{in } U$$

and touches u from above in U. This contradicts Theorem 3.1. □

Theorem 5.2. *If $u \in C^+([0,T) \times \mathbb{R}^d)$ and $v \in C^-([0,T) \times \mathbb{R}^d)$ are bounded and*

$$\begin{cases} D_t u - H(x, D_x u) \leq 0 & \text{in } (0,T) \times \mathbb{R}^d \\ D_t v - H(x, D_x v) \geq 0 & \text{in } (0,T) \times \mathbb{R}^d \\ u \leq v & \text{on } \{0\} \times \mathbb{R}^d, \end{cases}$$

then $u \leq v$ in $(0,T) \times \mathbb{R}^d$.

Proof. Suppose for contradiction that $\sup(u - v) > 0$.

Step 1. Using Lemma 3.2, the perturbation

$$v_\varepsilon(t,x) = v(t,x) + \varepsilon\sqrt{1 + |x|^2} + 2\omega(\varepsilon)\frac{Tt}{T-t}$$

satisfies

$$D_t v_\varepsilon - H(x, D_x v_\varepsilon) - \omega(\varepsilon) \geq 0 \quad \text{in } (0,T) \times \mathbb{R}^d.$$

Step 2. For small $\varepsilon > 0$,

$$\sup_{(0,T)\times\mathbb{R}^d} (u - v_\varepsilon) = \max_{(0,T)\times\mathbb{R}^d} (u - v_\varepsilon).$$

Thus, some vertical translation of v_ε (which satisfies the same differential inequality) touches u from above in $(0,T)\times\mathbb{R}^d$. This contradicts Theorem 3.1. □

Remark 5.3. In both of the above theorems, the boundedness requirement on u and v can be relaxed. The proofs need only that $u-v$ has sublinear growth at infinity.

6. Finite horizon games

We study two-player zero-sum differential games. In these games, two players compete over a payoff by controlling the dynamics of a particle in $\mathbb{R}^d$.

Consider a dynamics given by an ordinary differential equation

$$\mathbf{x}' = f(\mathbf{x},\mathbf{y},\mathbf{z}),$$

where $f:(\mathbb{R}^d)^3\to\mathbb{R}^d$ is bounded, uniformly continuous, and Lipschitz in its first argument. Our hypotheses on f guarantee unique solvability.

Theorem 6.1. *Given an initial state $x\in\mathbb{R}^d$, a time interval $[t,T)\subseteq\mathbb{R}$, and measurable control inputs $\mathbf{y},\mathbf{z}:[t,T)\to\mathbb{R}^d$, there is a unique Lipschitz continuous solution $\mathbf{x}:[t,T)\to\mathbb{R}^d$ of*

$$\begin{cases}\mathbf{x}'(s) = f(\mathbf{x}(s),\mathbf{y}(s),\mathbf{z}(s)) & \text{for } s\in[t,T)\\ \mathbf{x}(t)=x.\end{cases}$$

We call $\mathbf{x}=\mathbf{x}(\cdot;x,\mathbf{y},\mathbf{z})$ the system response. □

To define competition, we need a notion of strategy. The set of controls is

$$\mathcal{M}_{t,T} = \{\mathbf{y}:[t,T)\to\mathbb{R}^d \text{ measurable}\}.$$

The set of strategies is

$$\mathcal{N}_{t,T} = \{\beta:\mathcal{M}_{t,T}\to\mathcal{M}_{t,T} \text{ non-anticipating}\},$$

where non-anticipating means that

$$\mathbf{y}=\tilde{\mathbf{y}} \text{ a.e. in } [t,s) \quad \text{implies} \quad \beta[\mathbf{y}]=\beta[\tilde{\mathbf{y}}] \text{ a.e. in } [t,s)$$

for all $\mathbf{y},\tilde{\mathbf{y}}\in\mathcal{M}_{t,T}$ and $s\in[t,T]$.

A finite horizon game is specified by the dynamics above, together with a time interval $[0,T]$ and a bounded and Lipschitz function $g:\mathbb{R}^d\to\mathbb{R}$. Players I and II seek to maximize and minimize, respectively, the payoff $g(\mathbf{x}(T))$. To play, one player chooses a strategy and then the other player chooses a control in response. This leads to two value functions, depending on who chooses first. These are

$$V^+(t,x) = \inf_{\beta\in\mathcal{N}_{t,T}}\sup_{\mathbf{y}\in\mathcal{M}_{t,T}} g(\mathbf{x}(T;x,\mathbf{y},\beta[\mathbf{y}]))$$

and

$$V^-(t,x) = \sup_{\alpha\in\mathcal{N}_{t,T}}\inf_{\mathbf{z}\in\mathcal{M}_{t,T}} g(\mathbf{x}(T;x,\alpha[\mathbf{z}],\mathbf{z})),$$

for $t\in[0,T]$ and $x\in\mathbb{R}^d$.

A priori, we know that $V^+\geq V^-$. Our goal is to show that $V^+=V^-$. This implies there is no disadvantage to choosing first. This is related to the existence of a feedback control.

7. Dynamic programming

There are natural bijections

$$\mathcal{M}_{t,T} \cong \mathcal{M}_{t,s} \times \mathcal{M}_{s,T}$$

and

$$\mathcal{N}_{t,T} \cong \mathcal{N}_{t,s} \times (\mathcal{N}_{s,T})^{\mathcal{M}_{t,s}},$$

where A^B denotes the set of all functions from $B \to A$. Moreover,

$$\inf_{h \in A \to B} \sup_{a \in A} k(a, h(a)) = \sup_{a \in A} \inf_{b \in B} k(a, b),$$

holds for all sets A, B and functions $k : A \times B \to \mathbb{R}$. These observations allow us to divide the game into two phases.

Lemma 7.1. *The value functions satisfy*

$$V^+(t,x) = \inf_{\beta \in \mathcal{N}_{t,s}} \sup_{\mathbf{y} \in \mathcal{M}_{t,s}} V^+(s, \mathbf{x}(s; x, \mathbf{y}, \beta[\mathbf{y}]]))$$

and

$$V^-(t,x) = \sup_{\alpha \in \mathcal{N}_{t,s}} \inf_{\mathbf{z} \in \mathcal{M}_{t,s}} V^-(s, \mathbf{x}(s; x, \alpha[\mathbf{z}], \mathbf{y}))$$

for all $0 \le t \le s \le T$ *and* $x \in \mathbb{R}^d$.

Proof. Compute

$$\begin{aligned}
V^+(t,x) &= \inf_{\beta \in \mathcal{N}_{t,T}} \sup_{\mathbf{y} \in \mathcal{M}_{t,T}} g(\mathbf{x}(T; x, \mathbf{y}, \beta[\mathbf{y}])) \\
&= \inf_{\beta \in \mathcal{N}_{t,s}} \inf_{\tilde\beta : \mathcal{M}_{s,t} \to \mathcal{N}_{s,T}} \sup_{\tilde{\mathbf{y}} \in \mathcal{M}_{t,s}} \sup_{\mathbf{y} \in \mathcal{M}_{t,s}} g(\mathbf{x}(T; \mathbf{x}(s; x, \mathbf{y}, \beta[\mathbf{y}]), \tilde{\mathbf{y}}, \tilde\beta[\mathbf{y}][\tilde{\mathbf{y}}])) \\
&= \inf_{\beta \in \mathcal{N}_{t,s}} \sup_{\tilde{\mathbf{y}} \in \mathcal{M}_{t,s}} \inf_{\tilde\beta \in \mathcal{N}_{s,T}} \sup_{\mathbf{y} \in \mathcal{M}_{t,s}} g(\mathbf{x}(T; \mathbf{x}(s; x, \mathbf{y}, \beta[\mathbf{y}]), \tilde{\mathbf{y}}, \tilde\beta[\tilde{\mathbf{y}}])) \\
&= \inf_{\beta \in \mathcal{N}_{t,s}} \sup_{\mathbf{y} \in \mathcal{M}_{t,s}} V^+(s, \mathbf{x}(s; x, \mathbf{y}, \beta[\mathbf{y}])).
\end{aligned}$$

A similar computation holds for V^-. ☐

The dynamic programming principle suggests that V^+ and V^- solve a partial differential equation. Suppose for the moment that the value functions are smooth and the controls are piecewise constant. Hueristically, we compute

$$\begin{aligned}
0 &= \inf_{\beta \in \mathcal{N}_{t,s}} \sup_{\mathbf{y} \in \mathcal{M}_{t,s}} \frac{V^+(s, \mathbf{x}(s; x, \mathbf{y}, \beta[\mathbf{y}]])) - V^+(t,x)}{s-t} \\
&= \inf_z \sup_y \frac{V^+(s, x + (s-t) f(x,y,z) + o(s-t)) - V^+(t,x)}{s-t} \\
&= \inf_z \sup_y D_t V^+(t,x) + D_x V^+(t,x) \cdot f(x,y,z) + o(1)
\end{aligned}$$

as $s \to t^+$. In particular, we expect that V^+ solves

$$0 = D_t V^+ + H^+(x, D_x V^+),$$

where

$$H^+(x,p) = \inf_{z \in \mathbb{R}^d} \sup_{y \in \mathbb{R}^d} p \cdot f(x,y,z).$$

Simiarly, we expect that V^- solves

$$0 = D_t V^- + H^-(x, D_x V^-),$$

where

$$H^-(x,p) = \sup_{y \in \mathbb{R}^d} \inf_{z \in \mathbb{R}^d} p \cdot f(x,y,z).$$

We call H^+ and H^- the upper and lower Hamiltonians.

When players control different parts of the state, these are equal.

Lemma 7.2. *If $f(x,y,z) = f_1(x,y) + f_2(x,z)$, then $H^+ = H^-$.* □

We henceforth assume the min-max condition

$$H^+ = H^-,$$

and write $H = H^\pm$.

Lemma 7.3. *The game Hamiltonian H satisfies*

$$H(x,tp) = tH(x,p) \quad \text{for } t > 0,$$

$$|H(x,p)| \le C|p|,$$

and

$$|H(x,p) - H(\tilde{x},\tilde{p})| \le C(|x-\tilde{x}||p| + |p-\tilde{p}|).$$

Proof. This is immediate from the boundedness and uniform continuity of f. □

8. Uniqueness of value

We use viscosity solutions to show that $V^+ = V^-$.

Lemma 8.1. *The value functions $V^\pm$ are bounded and Lipschitz.*

Proof. Fix $\mathbf{y}, \mathbf{z} \in \mathcal{M}_{t,T}$ and consider what happens if we vary the initial state x and initial time t. By standard ODE stability, the map

$$(s,x) \mapsto \bar{x}(T; x, \mathbf{y}|_{[s,T)}, \mathbf{z}|_{[s,T)})$$

is Lipschitz on $[t,T] \times \mathbb{R}^d$, with constant depending only on f and $[t,T]$. Since g is bounded and Lipschitz, we conclude that $V^\pm$ is an inf/sup of uniformly bounded and uniformly Lipschitz functions. □

Formalizing the hueristic computation from before, we show that $V^\pm$ are equal to the unique viscosity solution of a Cauchy problem.

Theorem 8.2. *The time-reversal $v^\pm(t,x) = V^\pm(T-t,x)$ satisfies*

$$\begin{cases} D_t v^\pm - H(x, D_x v^\pm) = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ v^\pm = g & \text{on } \{0\} \times \mathbb{R}^d \end{cases}$$

In particular, $V^+ = V^-$.

Proof. Suppose $\varphi \in C^\infty((0,T)\times\mathbb{R}^d)$ touches V^+ from above at $(t,x) \in (0,T)\times\mathbb{R}^d$. For $s \in [t,T]$, the dynamic programming principle implies

$$\begin{aligned}
0 &= \inf_{\beta \in \mathcal{N}_{t,s}} \sup_{\mathbf{y} \in \mathcal{M}_{t,s}} V^+(s, \mathbf{x}(s;t,x,\mathbf{y},\beta[\mathbf{y}])) - V^+(t,x) \\
&\le \inf_{\beta \in \mathcal{N}_{t,s}} \sup_{\mathbf{y} \in \mathcal{M}_{t,s}} \varphi(s, \mathbf{x}(s;t,x,\mathbf{y},\beta[\mathbf{y}])) - \varphi(t,x) \\
&\le \inf_{z \in \mathbb{R}^d} \sup_{\mathbf{y} \in \mathcal{M}_{t,s}} \varphi(s, \mathbf{x}(s;t,x,\mathbf{y},z)) - \varphi(t,x) \\
&= \inf_{z \in \mathbb{R}^d} \sup_{\mathbf{y} \in \mathcal{M}_{t,s}} \int_t^s D_t\varphi(r, \mathbf{x}(r;t,x,\mathbf{y}(r),z)) \\
&\qquad\qquad + D_x\varphi(r, \mathbf{x}(r;t,x,\mathbf{y}(r),z)) \cdot \mathbf{x}'(r;t,x,\mathbf{y}(r),z))\,dr \\
&= \inf_{z \in \mathbb{R}^d} \sup_{\mathbf{y} \in \mathcal{M}_{t,s}} \int_t^s D_t\varphi(t,x) + D_x\varphi(t,x) \cdot f(x,\mathbf{y}(r),z) + o(1)\,dr \\
&= \inf_{z \in \mathbb{R}^d} \sup_{y \in \mathbb{R}^d} (s-t)\left[D_t\varphi(t,x) + D_x\varphi(t,x) \cdot f(x,y,z) + o(1)\right] \\
&= (s-t)\left[D_t\varphi(t,x) + H(x, D_x\varphi(t,x)) + o(1)\right].
\end{aligned}$$

We conclude that

$$D_t\varphi(t,x) + H(x, D_x\varphi(t,x)) \ge 0.$$

When φ touches V^+ from below at (t,x), we compute

$$\begin{aligned}
0 &= \inf_{\beta \in \mathcal{N}_{s,t}} \sup_{\mathbf{y} \in \mathcal{M}_{s,t}} V^+(s, \mathbf{x}(s;x,\mathbf{y},\beta[\mathbf{y}])) - V^+(t,x) \\
&\ge \inf_{\beta \in \mathcal{N}_{s,t}} \sup_{\mathbf{y} \in \mathcal{M}_{s,t}} \varphi(s, \mathbf{x}(s;x,\mathbf{y},\beta[\mathbf{y}])) - \varphi(t,x) \\
&\ge \sup_{\mathbf{y} \in \mathcal{M}_{s,t}} \inf_{\mathbf{z} \in \mathcal{M}_{s,t}} \varphi(s, \mathbf{x}(s;x,\mathbf{y},\mathbf{z})) - \varphi(t,x) \\
&\ge \sup_{y \in \mathbb{R}^d} \inf_{\mathbf{z} \in \mathcal{M}_{s,t}} \varphi(s, \mathbf{x}(s;x,y,\mathbf{z})) - \varphi(t,x)
\end{aligned}$$

and then proceed as before, concluding

$$D_t\varphi(t,x) + H(x, D_x\varphi(t,x)) \le 0.$$

Reversing time, we see that v^+ satisfies

$$D_t v^+ - H(x, D_x v^+) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d.$$

That v^+ is continuous on $([0,T)\times\mathbb{R}^d)$ and takes on the boundary conditions follows from the lemma above. The proof for v^- is symmetric. Uniqueness follows by comparison. □

9. Stability

Differential inequalities in the viscosity sense are stable with respect to many types of perturbations. This feature is as important as strict comparison.

Lemma 9.1. *If $u_1, u_2 \in C^+(U)$ satisfy $F(x,u_i,Du_i) \le 0$ in U, then $u = \max\{u_1,u_2\}$ satisfies $F(x,u,Du) \le 0$.*

Proof. If $\varphi \in C^1(U)$ touches u from above at $x \in U$, then it must touch either u_1 or u_2 from above at x. □

Definition 9.2. If $u : U \to \mathbb{R}$ is locally bounded, then

$$u^* = \min\{v \in C^+(U) : v \geq u\}$$

and

$$u_* = \max\{v \in C^-(U) : v \leq u\}$$

are its upper and lower semicontinuous envelopes.

Lemma 9.3. *Suppose $u_n \in C^+(U)$ satisfy and $F(x, u_n, Du_n) \leq 0$ and the pointwise limit $u = \lim_n u_n$ exists. If the sequence u_n is monotone or if the convergence of $u_n \to u$ is locally uniform, then $F(x, u^*, Du^*) \leq 0$.*

Proof. Suppose that $\varphi \in C^1(U)$ touches u^* from above at $x \in U$. Consider the perturbation

$$\psi(y) = \varphi(y) + |y - x|^4.$$

For small $r > 0$, we have

$$(u^* - \psi)(x) = 0 \quad \text{and} \quad \sup_{\bar{B}(x,2r) \setminus B(x,r)} (u^* - \psi) \leq -r^4.$$

We claim that, for large enough n,

$$\max_{\bar{B}(x,r)} (u_n - \psi) \geq -\frac{1}{3} r^4 \quad \text{and} \quad \sup_{\bar{B}(x,2r) \setminus B(x,r)} (u_n - \psi) \leq -\frac{2}{3} r^4.$$

In the case of uniform convergence or $u_{n+1} \geq u_n$, this is easy. When $u_{n+1} \leq u_n$, we use the upper semicontinuity of u_n. Observe, for each $y \in U$, that we can choose large n and $\delta > 0$ such that $u \leq u_n \leq u + \frac{1}{3} r^4$ in $B(y, \delta)$. By compactness, we see that $u \leq u_n \leq u + \frac{1}{3} r^4$ in $\bar{B}(x, 2r)$ for all large n.

From the claim, it follows that the difference $u_n - \varphi$ attains a local maximum at some $x_n \in U$ for all large n. Moreover, since $r > 0$ is arbitrary, we may assume that $x_n \to x$ and $(u_n - \psi)(x_n) \to 0$ as $n \to \infty$. The differential inequality for u_n gives $0 \geq F(x_n, u_n(x_n), D\psi(x_n))$. Sending $n \to \infty$ and using the continuity of F gives $0 \geq F(x, \psi(x), D\psi(x)) = F(x, \varphi(x), D\varphi(x))$. □

10. Existence

Using stability and comparison together, we can construct solutions. Fix $U \subseteq \mathbb{R}^d$ open, $\Gamma \subseteq \partial U$ closed, $F \in C(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d)$, and $g : \Gamma \to \mathbb{R}$. Let S^+ denote the set of all $u \in C^+(U \cup \Gamma)$ that satisfy

$$\begin{cases} F(x, u, Du) \leq 0 & \text{in } U \\ u \leq g & \text{on } \Gamma. \end{cases}$$

Similarly, let S^- denote the set of all $v \in C^-(U \cup \Gamma)$ that satisfy

$$\begin{cases} F(x, v, Dv) \geq 0 & \text{in } U \\ v \geq g & \text{on } \Gamma. \end{cases}$$

We assume S^+ and S^- are non-empty and that comparison holds:

$$\sup_{u \in S^+} u \leq \inf_{v \in S^-} v.$$

Lemma 10.1. *The function $\bar{u} = \sup_{u \in S^+} u$ satisfies*

$$F(x, \bar{u}^*, D\bar{u}^*) \leq 0 \quad \textit{and} \quad F(x, \bar{u}_*, D\bar{u}_*) \geq 0 \quad \textit{in } U.$$

Proof. The first statement is stability. For the second statement, assume for contradiction that $\varphi \in C^1(U)$ touches $\bar{u}_*$ from below at $x \in U$ and $F(x, \varphi(x), D\varphi(x)) < 0$. For $r > 0$ small, the perturbation

$$\psi(y) = \varphi(x) + \frac{1}{2}r^4 - |y - x|^4$$

satisfies

$$F(x, \psi, D\psi) \leq 0 \quad \text{in } B(x, 2r) \subseteq U.$$

Moreover, we also have

$$\min_{\bar{B}(x,r)} (\bar{u}_* - \psi)(x) \leq -\frac{1}{2}r^4, \quad \text{and} \quad \min_{\bar{B}(x,2r)-B(x,r)} (\bar{u}_* - \psi) \geq \frac{1}{2}r^4.$$

Since $\bar{B}(x, 2r)$ is compact, we can find $u \in S^+$ such that

$$u_* \leq \bar{u}_* \leq u_* + \frac{1}{4}r^4 \quad \text{on } \bar{B}(x, 2r).$$

Thus

$$\min_{\bar{B}(x,r)} (u_* - \psi)(x) \leq -\frac{1}{2}r^4, \quad \text{and} \quad \min_{\bar{B}(x,2r)-B(x,r)} (u_* - \psi) \geq \frac{1}{4}r^4.$$

Now consider

$$v(x) = \begin{cases} \max\{u(x), \varphi(x)\} & \text{if } x \in B(x, 2r) \\ u(x) & \text{if } x \in U \setminus \bar{B}(x, r). \end{cases}$$

Lemma 9.1 implies $v \in S^+$. Since $\min_{B(x,2r)}(\bar{u} - v) < 0$, we have contradicted the maximality of $\bar{u}$. □

Lemma 10.2. *If there is a* $\bar{u} \in S^+$ *satisfying* $\bar{u}_* = g$ *on* Γ, *then*

$$\sup_{u \in S^+} u = \min_{v \in S^-} v.$$

Similarly, if there is a $\bar{v} \in S^-$ *satisfying* $\bar{v}^* = g$ *on* Γ, *then*

$$\max_{u \in S^+} u = \inf_{v \in S^-} v.$$

Proof. Fix $\bar{u} \in S^+$ and $\bar{v} \in S^-$ and suppose that $\bar{u}_* = g$. (The case when $\bar{v}^* = g$ is symmetric.) Define

$$U_n = \{x \in U : \mathrm{dist}(x, \Gamma) > 1/n\},$$
$$w_n = \sup\{u \in S^+ : u = \bar{u} \text{ on } U \setminus U_n\},$$

and

$$w = \sup_n w_n.$$

By comparison, we have

$$\bar{u} \leq w \leq \sup_{u \in S^+} u \leq \inf_{v \in S^-} v \leq \bar{v}.$$

Thus, it suffices to show that $w_* \in S^-$. Since $w \geq \bar{u}$, we have $w_* \geq \bar{u}_* \geq g$ on Γ. Thus, it suffices to show that $F(x, w_*, Dw_*) \geq 0$ in U. By stability, it suffices to show that $F(x, (w_n)_*, D(w_n)_*) \geq 0$ in U_n. This is what the previous lemma shows when applied to the domain U_n. □

Theorem 10.3. *If there are $\bar{u} \in S^+$ and $\bar{v} \in S^-$ such that $\bar{u}_* = \bar{v}^* = g$ on Γ, then the boundary value problem*

$$\begin{cases} F(x, \bar{w}, D\bar{w}) = 0 & \text{in } U \\ \bar{w} = g & \text{on } \Gamma \end{cases}$$

has a unique bounded solution $w \in C(U \cup \Gamma)$.

Proof. Any element of $S^+ \cap S^- \subseteq C(\bar{U})$ is a solution. Uniqueness is immediate from comparison. □

We already used finite horizon games to contruct solutions of Cauchy problems. Using this new method, we can handle more general problems. (We could also consider more general games.)

Corollary 10.4. *Suppose $H \in C(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies*

$$|H(x,p) - H(y,q)| \leq \omega(|x-y||p| + |p-q|)$$

and

$$\sup |H(x,0)| < \infty.$$

For all bounded $g \in C(\mathbb{R}^d)$, the Cauchy problem

$$\begin{cases} D_t u - H(x, D_x u) = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ u = g & \text{on } \{0\} \times \mathbb{R}^d \end{cases}$$

has a unique bounded viscosity solution $u \in C([0,T) \times \mathbb{R}^d)$.

Proof. It is enough to construct $\bar{u} \in S^+$ and $\bar{v} \in S^-$ such that $\bar{u}_* = \bar{v}^* = g$ on $\{0\} \times \mathbb{R}^d$. We construct $\bar{v}$. The construction of $\bar{u}$ is similar. Since g is bounded, there are $\alpha_k, s_k \in \mathbb{R}$ and $x_k \in \mathbb{R}^d$ such that

$$g(x) = \inf_k (s_k + \alpha_k |x - x_k|) \quad \text{for } x \in \mathbb{R}^d.$$

Let

$$\beta_k = \sup_x \sup_{|p| \leq \alpha_k} |H(x,p)| \leq \sup_x |H(x,0)| + \omega(\alpha_k) < \infty.$$

It follows that

$$\varphi_k(t,x) = s_k + \alpha_k |x - x_k| + \beta_k t$$

satisfies

$$D_t \varphi_k - H(x, D_x \varphi_k) \geq 0 \quad \text{in } (0,\infty) \times \mathbb{R}^d.$$

Finally, observe that

$$\bar{v}(t,x) = \inf_k \varphi_k(t,x)$$

lies in $S^- \cap C([0,T) \times \mathbb{R}^d)$ and satisfies $\bar{v} = g$ on $\{0\} \times \mathbb{R}^d$. □

11. Minimum time games

We now consider a much more interesting class of games. We assume the same dynamics as before. The minimum time game has payoff

$$\tau(x, \mathbf{y}, \mathbf{z}) = \min\{t \geq 0 : \mathbf{x}(t; x, \mathbf{y}, \mathbf{z}) \in \Gamma\},$$

where $\Gamma \subseteq \mathbb{R}^d$ is closed and non-empty. The upper and lower value functions are

$$W^+(x) = \inf_{\beta \in \mathcal{N}_{0,\infty}} \sup_{\mathbf{y} \in \mathcal{M}_{0,\infty}} \tau(x, \mathbf{y}, \beta[\mathbf{y}]))$$

and

$$W^-(x) = \sup_{\alpha \in \mathcal{N}_{0,\infty}} \sup_{\mathbf{z} \in \mathcal{M}_{0,\infty}} \tau(x, \alpha[\mathbf{z}], \mathbf{z}),$$

for $x \in \mathbb{R}^d$. As before, we have $W^+ \geq W^-$ for free and hope to prove $W^+ = W^-$.

Example 11.1. If $f(x, y, z) = z/(1 + |z|)$, then this is the exit time problem we considered in the first lecture. Thus, provided $\Gamma = \mathbb{R}^d \setminus U$ with U bounded, we know that $W^+ = W^-$ is the unique solution of the Eikonal equation $|Dw| - 1 = 0$ in U with zero boundary conditions.

Lemma 11.2. *The minimum time value functions satisfy*

$$W^+(x) = t + \inf_{\beta \in \mathcal{N}_{0,t}} \sup_{\mathbf{y} \in \mathcal{M}_{0,t}} W^+(\mathbf{x}(t; x, \mathbf{y}, \beta[\mathbf{y}]]))$$

and

$$W^-(x) = t + \sup_{\alpha \in \mathcal{N}_{0,t}} \inf_{\mathbf{z} \in \mathcal{M}_{0,t}} W^-(\mathbf{x}(t; x, \alpha[\mathbf{z}], \mathbf{y}))$$

for all $x \in \mathbb{R}^d$ and $0 \leq t < \|f\|_{L^\infty}^{-1} \operatorname{dist}(x, \Gamma)$.

Proof. The hypothesis $0 \leq t < \|f\|_{L^\infty}^{-1} \operatorname{dist}(x, \Gamma)$ implies that

$$\bar{x}(t; x, \mathbf{y}, \mathbf{z}) \notin \Gamma$$

for all $\mathbf{y}, \mathbf{z} \in \mathcal{M}_{0,t}$. We compute

$$\begin{aligned}
W^+(x) &= \inf_{\beta \in \mathcal{N}_{0,\infty}} \sup_{\mathbf{y} \in \mathcal{M}_{0,\infty}} \tau(x, \mathbf{y}, \beta[\mathbf{y}]) \\
&= \inf_{\beta \in \mathcal{N}_{0,t}} \inf_{\tilde{\beta} : \mathcal{M}_{0,t} \to \mathcal{N}_{0,\infty}} \sup_{\mathbf{y} \in \mathcal{M}_{0,t}} \sup_{\tilde{\mathbf{y}} \in \mathcal{M}_{0,\infty}} t + \tau(\mathbf{x}(t; x, \mathbf{y}, \beta[\mathbf{y}]), \tilde{\mathbf{y}}, \tilde{\beta}[\mathbf{y}][\tilde{\mathbf{y}}]) \\
&= \inf_{\beta \in \mathcal{N}_{0,t}} \sup_{y \in \mathcal{M}_{0,t}} \inf_{\tilde{\beta} \in \mathcal{N}_{t,\infty}} \sup_{\tilde{\mathbf{y}} \in \mathcal{M}_{t,\infty}} t + \tau(\mathbf{x}(t; x, \mathbf{y}, \beta[\mathbf{y}]), \tilde{\mathbf{y}}, \tilde{\beta}[\mathbf{y}][\tilde{\mathbf{y}}]) \\
&= t + \inf_{\beta \in \mathcal{N}_{0,\infty}} \sup_{\mathbf{y} \in \mathcal{M}_{0,\infty}} W^+(\mathbf{x}(t; x, \mathbf{y}, \beta[\mathbf{y}]))
\end{aligned}$$

and similarly for V^-. □

Repeating the heuristic analysis of the minimum time case, we expect that $W^\pm$ both solve the boundary value problem

$$\begin{cases} -H(x, Dw) = 1 & \text{in } U \\ w = 0 & \text{on } \partial U. \end{cases}$$

This problem is more difficult to analyze than the time-dependent analogue we discused before. Indeed, the solutions may be unbounded or discontinuous.

Example 11.3. Suppose $\Gamma = \{(x,y) \in \mathbb{R}^2 : x \leq \lfloor y \rfloor\}$ and $f(x,y,z) = (-1,0)$. Then $W^{\pm}(x,y) = \max\{0, x - \lfloor y \rfloor\}$ is both unbounded and discontinuous. Note that $W^{\pm}$ is lower semicontinuous.

Boundedness is obtained by transformation. Observe, if $\varphi \in C^1(\mathbb{R}^d)$ and $\psi = 1 - e^{-\varphi}$, then $-H(x, D\varphi) = 1$ is equivalent to $\psi - H(x, D\psi) = 1$. Since the map $s \mapsto 1 - e^{-s}$ is smooth and increasing, we expect this formal computation to hold in the sense of viscosity.

Lemma 11.4. *If $w = 1 - \exp(-W^{\pm})$, then*

$$w^* - H(x, Dw^*) \leq 1 \quad \text{and} \quad w_* - H(x, Dw_*) \geq 1 \quad \text{in } U.$$

Proof. Exercise. □

We hope that $1 - \exp(-W^{\pm})$ are both the unique solution of

$$\begin{cases} w - H(x, Dw) = 1 & \text{in } U \\ w = 0 & \text{on } \partial U. \end{cases} \tag{11.1}$$

Obseve that the constant functions 0 and 1 are sub and supersolutions. Since 0 takes on the boundary conditions, we can apply Lemma 10.2 to conclude that

$$\sup_{u \in S^+} u = \min_{v \in S^-} v,$$

where $S^{\pm}$ are defined as in Section 10. Constructing a supersolution that takes on the boundary value is much harder. Still, this unique minimal supersolution is the natural candidate for the solution.

12. Regularity and control

In the presence of additional structure, bounded subsolutions are automatically more regular.

Lemma 12.1. *Suppose $U \subseteq \mathbb{R}^d$ is open and $H \in C(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies*

$$H(x,p) \geq c|p| - C.$$

If $u \in C^+(U)$ is bounded

$$u + H(x, Du) \leq 0 \quad \text{in } U,$$

then u is Lipschitz.

Proof. Suppose that $|u| \leq \alpha$ and choose $\beta > 0$ such that

$$H(x,p) \geq \alpha + 1 \quad \text{for } |p| \geq \beta.$$

Suppose $B(x,r) \subseteq U$ and consider the function

$$\psi_\varepsilon(y) = u(x) + \beta|y - x| + \frac{\varepsilon}{r - |y - x|}.$$

Observe that

$$\psi_\varepsilon + H(x, D\psi_\varepsilon) \geq 1 \quad \text{in } \mathbb{R}^d \setminus \{0\}.$$

Since

$$\psi_\varepsilon \geq u \quad \text{on } \{x\} \cup \partial B(x,r),$$

comparison implies that $u \leq \psi_\varepsilon$ in $B(x,r)$. It follows that $u(y) - u(x) \leq \beta|y - x|$ whenever $B(x, |y-x|) \subseteq \mathbb{R}^d$. Thus u is Lipschitz. □

This assumption also allows us to solve the minimum time problem.

Corollary 12.2. *If $U \subseteq \mathbb{R}^d$ is open and $H \in C(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies*

$$|H(x,p) - H(y,q)| \leq \omega(|x-y||p| + |p-q|)$$

and

$$H(x,p) \geq c|p| - C,$$

then

$$\begin{cases} u + H(x,Du) = 1 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

has a unique solution $u \in C(\bar{U})$.

Proof. Note that 0 is a subsolution taking on the boundary values. Moreover, the coercivity guarantees that $v(x) = \alpha \operatorname{dist}(x, \partial U)$ for $\alpha > 0$ large is a supersolution taking on the boundary values. Apply Perron's method to obtain a solution. Uniqueness follows by comparison. □

The coercivity condition $H(x,p) \geq c|p| - C$ is sometimes called controllability. In the context of differential games, it means that player II can force the velocity to have positive inner product with any given unit vector.

Corollary 12.3. *The value functions $w = 1 - \exp(-W^{\pm})$ are each the unique solution of*

$$\begin{cases} w - H(x,Dw) = 1 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

when the game Hamiltonian satisfies $H(x,p) \leq -c|p|$.

Proof. Exercise. □

13. Monotone and consistent approximation

A variation on our proof of stability gives convergence of certain finite difference approximations. This is useful if one wishes to numerically approximate viscosity solutions. It is also useful in capturing scaling limits of certain discrete processes.

Consider a partial differential equation

$$F(x,u,Du) = 0 \quad \text{in } U$$

where $U \subseteq \mathbb{R}^d$ open and $F \in C(U \times \mathbb{R} \times \mathbb{R}^d)$.

Our approximation schemes consist of a sequence pairs

$$(X_n, M_n)$$

where $X_n \subseteq \mathbb{R}^d$ is locally finite and $M_n : \mathbb{R}^{X_n} \to \mathbb{R}_{X_n \cap U}$. We make the following assumptions:

(1) Density: There is a sequence $\delta_n \downarrow 0$ and maps $\pi_n : U \to X_n \cap U$ such that $|\pi_n(x) - x| \leq \delta_n$ for all $x \in U$.
(2) Locality: There is a sequence $\varepsilon_n \downarrow 0$ such that, if $x \in X_n \cap U$ and $\varphi, \psi \in \mathbb{R}^{X_n}$ agree on $B(x, \varepsilon_n) \cap X_n$, then $M_n\varphi(x) = M_n\psi(x)$.
(3) Monotonicity: If $\varphi : X_n \to \mathbb{R}$ touches $\psi : X_n \to \mathbb{R}$ from above at $x \in X_n \cap U$, then $M_n\varphi(x) \leq M_n\psi(x)$.
(4) Consistency: If $\varphi \in C^1(U)$, $x \in U$, $s_n \in \mathbb{R}$, $x_n \in X_n$, $\lim_{n\to\infty} s_n = 0$, and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} M_n(\varphi + s_n)(x_n) = F(x, \varphi(x), D\varphi(x))$.

Example 13.1. If $U = (0,1)^d$ and $X_n = \{0/n, 1/n, ..., n/n\}^d$, then

$$M_n u(x) = n(u(x) - u(x - e_k/n))$$

approximates $F(x, s, p) = p_k$ in $U = (0,1)^d$.

Theorem 13.2. *If $u_n : X_n \to \mathbb{R}$ satisfy $M_n u_n \le 0$ and $\bar{u}_n = u_n \circ \pi_n \to u \in C(U)$ locally uniformly as $n \to \infty$, then $F(x, u, Du) \le 0$ in U.*

Proof. Suppose that $\varphi \in C^1(U)$ touches u from above at $x \in U$. We may assume that $x = 0$ and $\bar{B}(0,1) \subseteq U$. Consider the perturbation

$$\psi(x) = \varphi(x) + |x|^4.$$

For $r \in (0,1)$, we have

$$\max_{\bar{B}(0,r)} (u - \psi) = 0 \quad \text{and} \quad \max_{\bar{B}(0,1) \setminus B(0,r)} (u - \psi) \le -r^4.$$

Since the convergence is uniform on the compact ball $\bar{B}(0,1)$, we see that

$$\max_{\bar{B}(0,r)} (\bar{u}_n - \psi) \ge -\frac{1}{3} r^4 \quad \text{and} \quad \max_{\bar{B}(0,1) \setminus B(0,r)} (\bar{u}_n - \psi) \le -\frac{2}{3} r^4.$$

holds for all $n \ge n_r$. Thus $u_n - \psi$ attains its maximum on $B(0, 1 - \delta_n) \cap X_n$ at $x_n \in \bar{B}(x, r + \delta_n) \cap X_n$. Thus $\psi + s_n$, where $s_n = (u_n - \psi)(x_n)$, touches u_n from aboave at x_n in $\bar{B}(0,1) \cap X_n$. Since $r \in (0,1)$ is arbitary, we see that $x_n \to 0$ and $s_n = (u_n - \psi)(x_n) \to 0$ as $n \to \infty$. Using locality and monotonicity, we compute

$$0 \ge M_n u_h(x_n) \ge M_n(\psi + s_n)(x_n)$$

for all large n. We then compute

$$0 \ge \lim_{n \to \infty} M_n(\psi + s_n)(x_n) = F(0, \psi(0), D\psi(0)) = F(0, \varphi(0), D\varphi(0))$$

using consistency and the definition of ψ. ☐