(1) Suppose $U \subsetneq \mathbb{R}^d$ is open. Prove that the distance function

$$u(x) = \min_{y \in \mathbb{R}^d \setminus U} |x - y|$$

is the unique non-negative continuous function on $\mathbb{R}^d$ that satisfies

$$\begin{cases} \lim_{r \to 0} \frac{1}{r} \left( u(x) - \min_{\partial B(x,r)} u \right) = 1 & \text{if } x \in U \\ u(x) = 0 & \text{if } x \in \mathbb{R}^d \setminus U. \end{cases}$$

(2) Give a direct proof of the fact that $u \in C(U)$ satisfies $|Du| \geq 1$ in the sense of viscosity if and only if the following property holds. If $V \subseteq U$ is open and bounded and $\varphi \in C^\infty(U)$ satisfies

$$\begin{cases} |D\varphi| \leq 1 & \text{in } V \\ \varphi \leq u & \text{on } \partial V, \end{cases}$$

then $\varphi \leq u$ on $\bar{V}$.

(3) Suppose $U \subseteq \mathbb{R}^d$ is open and bounded. Give a simple proof that any viscosity solution of

$$\begin{cases} |Du| = 1 & \text{in } U \\ u = 0 & \text{on } \mathbb{R}^d \setminus U, \end{cases}$$

must be the distance function. Hint: convert the representation formula $u(x) = \min_{y \in \mathbb{R}^d \setminus U} |x - y|$ into two families of test functions that constrain the viscosity solution from above and below.

(4) Suppose $u \in C(U)$ and $x \in U$. Show that $p \in D^+u(x)$ holds if and only if there is a $\varphi \in C^1(U)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local maximum at $x$.