Research Statement

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In its simplest form, algebraic geometry studies objects which arise as the zero sets of polynomials. It is a broad subject, having been generalized considerably in the 20th century using the language of schemes, which intersects many other fields of mathematics like number theory and analytic geometry. I have explored only a few of its many facets, usually concentrating on problems that can be reduced to algebraic combinatorics. I am extremely fortunate to have worked on many fruitful projects with various mathematicians leading to 6 papers, 4 published and 2 being prepared for submission.

At the heart of these projects have been themes from birational geometry where we wish to represent a birational equivalence class by some well-behaved variety. Many of my projects have sought to find smooth representatives for example, often dealing with some form of resolution of singularities. When trying to find other convenient representatives, we might look for a variety which is Cohen-Macaulay, Gorenstein or perhaps a variety with specific singularities (such as those that can be eliminated using normalization).

My earliest research projects dealt with topics related to the minimal model program. My work with Kalle Karu nearly 8 years ago at the University of British Columbia concerned the factorization of birational maps between toric varieties into a sequence of blow-up and blow-down maps. This project on Oda's Strong Conjecture resulted in my first publication [9]. My subsequent work with Edward Bierstone and Pierre Milman at the University of Toronto produced a more specialized version of the characteristic 0 desingularization algorithm which detected and avoided simple normal crossings in the resolution process. This led to a publication [4], concluding my undergraduate research experience as I moved on to Cornell University for my PhD studies.

My most recent projects have included a number of proposed problems from my advisor Allen Knutson. For example, knowing my background in the resolution of singularities, he suggested that I try to apply techniques that I previously learned to Frobenius split varieties. The hope was that characteristic 0 methods could be used in these well-behaved positive characteristic varieties. This idea can be realized in low dimensions, highlighted in [7], providing a desingularization for certain F-pure singularities.

I have also worked on projects involving Schubert varieties, an important class of varieties for algebraic combinatorists. It is well known that a Schubert variety can be desingularized using a Bott-Samelson resolution; it does not however do this strictly (it is not a bijection over the smooth locus). On the other hand, Hironaka's result tells us that such varieties can be desingularized strictly. My work in [6] compares these two very different approaches to end up with a hybrid desingularization process which is strict as in Hironaka's method, yet similar to the Bott-Samelson construction.

My current work has dealt with understanding what happens when we blow-up the boundary divisor of a Schubert variety (or locally, the Kazhdan-Lusztig variety). The boundary divisor is actually an anticanonical divisor, making it useful to test for the Gorenstein property. I have shown that it takes only one blow-up along the boundary divisor to get a Gorenstein variety [8], providing a convenient (almost canonical) description for a Gorensteinization of Schubert varieties.

Lastly, I would like to mention my work on a group project in representation theory with Dan Barbasch. We studied the orbits of the G-diagonal action on multiple flag varieties in the exceptional cases. Results for the F_4 case were featured in [1], and work is ongoing for the E_6, E_7, E_8 cases.

While I have thought about problems involving the minimal model program (resolution of singularities, Frobenius splittings, factoring birational maps etc.) or algebraic combinatorics (Schubert varieties, toric varieties etc.), I look forward to the possibility of working with experts in other areas of algebraic geometry. A more detailed description of my work is detailed below. Many of these projects provide multiple opportunities for further future research, and I discuss specific ideas in each section.

Blowing-up Schubert varieties along their boundary

Schubert varieties are a well-studied class of varieties that have useful descriptions which reduce many otherwise difficult operations to simple combinatorics. Let G be a simple complex Lie group, and fix a maximal torus T as well as a Borel subgroup B containing T. By the Bruhat decomposition, B acts on G/B with finitely many orbits indexed by W, where $W = N_G(T)/T$ is the Weyl group of G. The closures of these orbits $X^w := \overline{BwB/B}$ are called Schubert varieties.

We can define the boundary of X^w as $\mathrm{bdry}(X^w) = \bigcup_{w \lessdot v} X^v$, using v which cover w in strong Bruhat order. There is a Frobenius splitting on X^w for which $\mathrm{bdry}(X^w)$ is compatibly split. It turns out that $\mathrm{bdry}(X^w)$ is an anticanonical divisor which makes it a useful tool for talking about the Gorenstein property (recall that a variety is Gorenstein if its anticanonical divisor is Cartier). The question is, what happens when we blow-up this boundary divisor?

On a local level, the question reduces to blowing-up a Kazhdan-Lusztig variety along its boundary. In [11], it was shown that there exists a degeneration of the coordinate ring of a Kazhdan-Lusztig variety to that of a Stanley-Reisner ideal of a subword complex (which can be viewed as a toric variety). Subword complexes were introduced in [12] and provide a geometric way of viewing subwords of a word written in elements of Coxeter group.

I show that we can choose this degeneration so that it commutes with blowing-up along the boundary divisor. Furthermore, since the blow-up of the degeneration has an exceptional divisor as its boundary divisor, it is Gorenstein. We use the Frobenius splitting induced from the degeneration to define the boundary of the blow-up and show that it is anticanonical. Since being Gorenstein is open in flat families, the blow-up of the Kazhdan-Lusztig variety is also Gorenstein.

We can better understand the blow-up algebra using syzygies. I introduce the idea of a syzygy ideal for this purpose, allowing us to extend the Frobenius splitting to the blow-up. Through a series of isomorphisms, we can embed the total transform into a simple GIT quotient of a product of Kazhdan-Lusztig varieties.

In summary, I show that the blow-up of a Schubert variety along its boundary is Gorenstein, and that this blow-up can be viewed as a similar combinatorial object to the original Schubert variety.

An extension of this problem involves the involution subword complex which attempts to generalize the idea of a subword complex (involving orbits of a B-action on G/B) to a complex related to an involution stable Borel acting on G/K. Such a description of the Gröbner geometry of B-orbits on G/K would allow for the above techniques to be applied in this more general setting.

Resolution of Singularities: Positive characteristic

Can Frobenius splittings be used to resolve singularities? The idea is that Frobenius split varieties are well-behaved enough to allow for a choice of successive centers of blowing-up leading to a smooth variety. A Frobenius splitting on a variety X can be defined using a section of the anticanonical bundle ω_X^{-1} of the variety. A first important question is whether one can choose an appropriate section of the anticanonical bundle that defines a Frobenius splitting on the strict transform (this way, we have a control on the variety during the blowing-up process, and strict transforms of split varieties are split). I have made progress towards using the characteristic 0 desingularization algorithm in this positive characteristic setting, using centers whose codimension satisfies a certain bound. This bound would allow for smoothness to be achieved for curves and surfaces, and could provide an algorithm to simplify these F-pure singularities in higher dimensions. A desingularization of the compatibly split varieties is also possible.

Preliminary results have been typed up in preparation for publication. I can show that hypersurfaces of dimensions 1 and 2 defining Frobenius splittings can be desingularized using the characteristic 0 algorithm (in other words, using hypersurfaces of maximal contact). Such techniques have long been sought after in characteristic p, and while my result does not solve this problem in general, it provides a rich set of examples to work with. I expect that my results can be improved with a better understanding of how Frobenius splittings play a role in desingularization.

Frobenius splittings in general have not been widely studied, and further investigation into their uses can prove beneficial for this purpose. For example, can we choose a Frobenius split variety as a representative for birational equivalence classes in a meaningful way (with actual morphisms as in the resolution of singularities)? If so, my result provides a step forward in finding convenient desingularizations in characteristic p, by first reducing the problem to investigating Frobenius split varieties, and then by applying specialized results to get smoothness.

Resolution of Singularities: Characteristic 0

Strict Bott-Samelson resolutions: In 1958, Bott and Samelson introduced certain spaces [5] that provided convenient desingularizations of Schubert varieties (which were later generalized by Hansen and Demazure). Around the same time, Hironaka published his famous result on the resolution of singularities for algebraic varieties in characteristic zero [10]. His result could of course be applied to Schubert varieties to obtain a second, very different sort of desingularization. While Hironaka's method is a stronger result in general, the fact that Bott-Samelson resolutions are the preferred desingularization method would speak to the combinatorial conveniences they possess. I wanted to find out if we could get the best of both methods.

An important feature of a Hironaka desingularization (such as the algorithm in [2]) is the fact that the desingularization map is an isomorphism over the smooth locus. One calls such a desingularization a strict resolution of singularities. In general, the Bott-Samelson resolution is not an isomorphism over the smooth locus of a Schubert variety. Even more, while Hironaka's method utilizes blow-ups, a Bott-Samelson resolution is not in general a blow-up map. Nevertheless, the resolution has many combinatorial properties that are natural for working with Schubert varieties. There is for example an action of a torus T with isolated fixed points, and the map on T-fixed points is especially easy to utilize.

I explore both desingularization techniques in more detail in [6], and suggest a more general method for resolving singularities which is close to the Bott-Samelson construction, and yet is a strict desingularization. Computer analysis provided an answer for low-dimensional examples, but more work is needed to generalize these results. The generalized Bott-Samelson varieties I used dealt with divisions of a word written in simple reflections. The fact that singularities can be detected using pattern embeddings and not subword conditions posed a problem in proving more general results. The use of subword complexes might serve as an especially useful tool for geometrically visualizing the subwords in consideration. While I discuss which Schubert varieties can be desingularized strictly, I do not provide a constructive method for doing so. I would be more satisfied with an explicit algorithm for doing this in simple examples that has the potential of being generalized to all Schubert varieties. This would provide a strengthened form of Bott-Samelson resolutions.

A desingularization algorithm which avoids simple normal crossings: In the usual resolution algorithm [2], we are given an algebraic variety which we would like to desingularize. This desingularization is given by a finite sequence of blowings-up with smooth centers determined by a resolution invariant. The result of the desingularization is a divisor with simple normal crossings. From this point of view, it is reasonable to ask if it is possible to preserve all simple normal crossing points that arise at any stage of the desingularization. In my group work with Edward Bierstone and Pierre Milman, we started by investigating if simple normal crossing points could be characterized by the invariant and perhaps some simple geometric information. By computing the invariant for a simple normal crossings point (in an arbitrary year of the resolution), we then asked if a point with this value of the invariant was in fact simple normal crossings. This was asking for too much, but we did show that such points could be written in a special local form. Using this local form, we altered the resolution algorithm to avoid all simple normal crossing points that may arise in the desingularization (see [4]).

Continuing with this idea of using the desingularization invariant as a tool to detect (and classify) certain singularities, the next natural question is whether we can alter the algorithm to avoid all normal crossings (that is simple normal crossings in an étale neighborhood, such as in a nodal curve singularity). The Whitney umbrella provides a definitive no to this question unless we include pinch points in this list of singularities to avoid. The question of which singularities should be grouped together is an interesting area of study (see [3] for more information). Answering such questions would give greater insight into the nature of certain singularities as well as provide tools for working with birational equivalence representatives other than smooth varieties.

Oda's strong conjecture

The question of the strong factorization of toric varieties (referred to as Oda's Strong Conjecture) is an important one in birational geometry. Roughly put, it asks if a birational map between two toric varieties can be factored into a sequence of blowings-up followed by a sequence of blowings-down. In my work with Kalle Karu, we attempted to develop an algorithm which constructed a strong factorization from a weak factorization. A weak factorization allows the blowings-up and blowings-down in any order (it is known to exist). The goal of the project was to show that this algorithm terminated. Because of the combinatorial nature of toric varieties, the local version of this algorithm has a convenient symbolic representation. This algorithmic approach, which was also considered earlier by Abramovich, Matsuki, and Wlodarczyk, simplifies the difficult combinatorial and geometric problem into one involving sequences of elementary matrices and a list of commutativity rules. Reducing this difficult problem to a convenient combinatorial algorithm was an important first step as it is very difficult to find a reasonable approach to the problem.

The result we found in [9] provided an important step in proving the strong factorization conjecture. The likely path to completing this result is to define some local measure of how complicated these sequences can get. Subsequently finding a bound for the measure would show that the algorithm terminates in finite time. I recently noticed that techniques from geometric group theory have not yet been applied to this problem. This is an area of mathematics which would be interesting to learn more about and I believe such approaches can provide the missing link to finally proving this conjecture.

Multiple flag varieties

A multiple flag variety is a product of flag varieties $G/P_1 \times ... \times G/P_n$, where each term in the product is the quotient of a reductive Lie group G with a parabolic subgroup P_i. There is a natural diagonal G-action on this product, and the question is, when are there finitely many orbits? This question was answered for the classical Lie groups (see [13]), but remained unsolved for the exceptional F_4, E_6, E_7, E_8 cases. Progress on this project has led to an explicit algorithmic technique for computing "canonical" orbit representatives. In my group work with Dan Barbasch, a classification for the F_4 case has been completed and featured in [1]. The work for the E_6, E_7 and E_8 cases is continuing. Such questions have an important role in representation theory and can provide tools for solving more general questions (involving actions of compact subgroups K for example).

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