

Strict Local Martingales, Bubbles, and No Early Exercise

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November 6, 2007

Abstract

We show pathological behavior of asset price processes modeled by continuous strict local martingales under a risk-neutral measure. The inspiration comes from recent results on financial bubbles. We analyze, in particular, the effect of the strict nature of the local martingale on the usual formula for the price of a European call option, especially a strong anomaly when call prices decay monotonically with maturity. A complete and detailed analysis for the archetypical strict local martingale, the reciprocal of a three dimensional Bessel process, has been provided. Our main tool is based on a general *h-transform* technique (due to Delbaen and Schachermayer) to generate positive strict local martingales. This gives the basis for a statistical test to verify a suspected bubble is indeed one (or not).

Introduction

Local martingales date back to the paper of K. Itô and S. Watanabe [14] in 1965, when they wanted a decomposition of multiplicative functionals within the theory of Markov processes. Quickly thereafter P.A. Meyer used local martingales for the general case of his famous Doob-Meyer decomposition theorem of submartingales. Local martingales emerged in Finance Theory when Kreps and later Delbaen and Schachermayer [4] and [6] revealed a central role played by local martingales in the First Fundamental Theorem of Asset Pricing (i.e., the mathematical conditions for the absence of arbitrage).

*Supported in part by NSF grant DMS-0306194; Mathematics Department and ORIE – 506 Malott Hall, Cornell University, Ithaca, NY, 14853-4201 USA

†Supported in part by NSF grant DMS-0202958 and NSA grant MDA-904-03-1-0092; ORIE – 219 Rhodes Hall, Cornell University, Ithaca, NY 14853-3801 USA

A local martingale is usually defined in an inclusive manner: it includes all martingales as special cases. To distinguish local martingales which are not regular enough to be actual martingales, we use in this paper the notion of *strict local martingales*, in keeping with the language of (for example) Delbaen-Schachermayer [5] and Elworthy-Li-Yor [8].

A natural question is: what happens to a financial market when the no arbitrage condition yields a strict local martingale (rather than a true martingale) under a risk neutral measure? Several authors have looked at this problem and offered solutions to anomalies which might result from the lack of the martingale property. One interesting perspective offered in this direction is the theory of price bubbles as argued in 2000 by Loewenstein and Willard [20]. They propose that to identify a bubble one needs to look at the difference between the market price of an asset and its fundamental price. Their argument is later complemented and further developed by Cox and Hobson [3] and the two articles by Jarrow, Protter, and Shimbo [16], [17]. Please see the latter articles for the definitions of the market and the fundamental prices of an asset and any of the other financial terms that follow. In particular, the authors in [16] and [17] classify bubbles into three types in an arbitrage-free market satisfying Merton's *No Dominance* condition (see [16] or [22]). One, in which the difference between the two price processes under an equivalent local martingale measure is a uniformly integrable martingale; two, when it is a martingale but non-uniformly integrable; and last, when it is a strict local martingale. In a static market with infinite horizon, for a stock which pays no dividends, Example 5.4 in [17] shows that the difference between the two prices is actually the current market price of the stock. Thus a stock price which behaves as a strict local martingale under an equivalent local martingale measure is an example of a price bubble of the third kind. Cox and Hobson [3], too, use this definition of stock price bubbles. They further furnish several interesting examples of bubbles both where volatility increases with price levels, and where the bubble is the result of a feedback mechanism. They go on to exhibit (among other things) how in the presence of bubbles put-call parity might not hold and call prices do not tend to zero as strike tends to infinity.

We mention in passing that in the Economics literature the study of bubbles is older. For example, the possibility that bubbles can exist in discrete-time infinite-horizon economies has been explored in Diba and Grossman [7], Kocherlakota [24], and Tirole [26]. In fact, it has only recently been studied using the tools of mathematical finance. In this vein, other than the ones already mentioned, see, for example, Jarrow and Madan [15], Gilles [10], Gilles and Leroy [11], and Huang and Werner [13].

The purpose of this paper is natural in this context. We consider a market with a single risky asset (stock) and zero spot interest rate. Let $\{S_t\}$, $t \in (0, \infty)$, be a positive continuous strict local martingale which models the discounted price of the (non-dividend paying) stock under an equivalent local martingale measure. Suppose one prices European options on this stock maturing at time T by the usual formula $E(S_T - K)^+$, then, it turns out that in many cases the price of a European call decreases *monotonically* as the maturity time increases after some time point. One should note the distinction from the case of true martingales for which the price curve strictly increases by Jensen's inequality. This, in turn, has interesting implications for American options. A standard truism for financial markets is Merton's *no early exercise* theorem which states that the prices of American calls and European calls are the same (given identical strike prices, maturity times, and the absence of dividends), which in turn implies that the optimal exercise time of an American call is at maturity (hence "no early exercise"). This famous theorem can be found in many textbooks, for example in that of Shreve [25], p. 363. The proof of this result however uses that the risky asset price is a (true) martingale under the risk neutral measure. If it is in fact a strict local martingale, then "no early exercise" need not hold, as shown in both [3] and [16], [17]. However, once we establish that the prices of European options decrease with maturity, it readily follows that "no early exercise" cannot hold. This feature may seem strange at first glance, but if we assume the existence of a financial bubble, the intuition is that it is advantageous to purchase a call with a short expiration time, since at the beginning of a bubble prices rise, sometimes dramatically. However in the long run it is disadvantageous to have a call, increasingly so as time increases, since the likelihood of a crash in the bubble taking place increases with time.

Of course, pricing a European option by the usual formula when the underlying asset price is a strict local martingale is itself controversial. For example, Heston, Loewenstein, and Willard [12] observe that under the existence of bubbles in the underlying price process, put-call parity might not hold, American calls have no optimal exercise policy, and lookback calls have infinite value. In a beautiful paper Madan and Yor [21] have recently argued that when the underlying price process is a strict local martingale, the price of a European call option with strike rate K should be modified as

$$\lim_{n \rightarrow \infty} E[(S_{T \wedge T_n} - K)^+],$$

where $T_n = \inf\{t \geq 0 : S_t \geq n\}$, $n \in \mathbb{N}$, is a sequence of hitting times.

From a purely mathematical point of view this analysis fits in the attempt towards a greater understanding of the untamed beasts that strict local martingales are. Although their appearance in the context of stochastic integration and martingale theory is natural, they still seem to elude manipulatability. If we restrict ourselves to nonnegative local martingales, it is known that they are also supermartingales (see [19, page 36]), and the decreasing property of $E(S_T - K)^+$ follows for $K = 0$. However, it seems that for a positive K , the problem is far from being trivial. Part of the reason is that Tanaka's formula ([19, page 203]) applied to the process $(S_T - K)^+$ does not result in any martingales. To avoid this problem, Madan and Yor [21] develop a modification of the usual Tanaka formula to obtain the right martingale in the decomposition. We take a different route, one which was discovered in a short and elegant article by Delbaen and Schachermayer [5]. While discussing surprising arbitrage possibilities in Bessel processes, these authors describe an effective way to generate strict local martingales which are positive throughout. The method is a mimic of an identity in law for the canonical example of a strict local martingale: the 1963 Class D counterexample of Johnson and Helms [18], known more commonly today as the *inverse Bessel process*. (A nice exposition of this process, including why it is a strict local martingale and not a true martingale, can be found in the book of Chung and Williams [2].) Consider a Brownian motion starting from one. If one changes the Wiener measure using as Radon-Nikodým derivative the martingale given by the Brownian motion absorbed upon hitting zero, the coordinate process has the law of a three-dimensional Bessel process (BES(3)). This change of measure technique is an example of Doob's *h-transform*. Delbaen and Schachermayer show that the reciprocal of all positive local martingales can in fact be obtained by a similar procedure, i.e., transforming the law of a nonnegative martingale by the process itself. We modify the result slightly in Lemma 4 to suit our purpose. Our analysis relies heavily on this machinery by transferring the problem about a strict local martingale to a problem about a true martingale.

The content of the paper is as follows. In the next section we do a complete analysis of the function $E(S_T - K)^+$ when S is the reciprocal of the three-dimensional Bessel process. Because of its relation to Brownian motion we are able to obtain precise and detailed results. Fix a strike price K and we study the prices of European calls as expiration time varies. That is, we look at the function $T \rightarrow E\{(S_T - K)^+\}$ with K fixed. We show that this function increases monotonically initially until a special time Ξ , and after Ξ it decreases monotonically. Moreover we find a sharp bound on the critical time Ξ . Next we look at the more general case of continuous

local martingales. Here we are unable to prove exactly analogous results to the case of the inverse Bessel process partly because they are not universal even among the class of positive strict local martingales. However we do achieve two important objectives. First, under a Markovian assumption, we establish sufficient conditions on the law of a local martingale process S under which $E\{(S_T - K)^+\}$ is decreasing in T for some large T onwards. This is the content of Proposition 7. Second, in Section 3 we demonstrate a counterexample of a strict local martingale process for which the monotone decreasing behavior of $T \mapsto E\{(S_T - K)^+\}$ does not hold for any K . However, for any general strict local martingale under suitable assumptions what we can easily show is that the prices of European options do indeed decrease, albeit in a more complicated manner:

Proposition 1. *Let $S_t, t \in (0, \infty)$, be a strictly positive, strict local martingale such that $S_0 = 1$, and $\lim_{t \rightarrow \infty} E(S_t) = 0$. Then, for any $K > 0$, and any time point $u \in (0, \infty)$, such that*

$$E\{(S_u - K)^+\} > 0,$$

there exists a time t_0 (depending on K and u) such that for all $t \geq t_0$, we have $E\{(S_t - K)^+\} < E\{(S_u - K)^+\}$.

This is, in essence, the same pathological behavior that the inverse Bessel process exhibits.

Finally we note in passing that this may provide a statistical test to verify if a suspected bubble is in fact a bubble, in the case of an absence of dividends, by checking the prices of European calls for short and long time horizons. Since the prices of American calls must be non-decreasing in all cases, so also will the prices of European calls be non-decreasing in the martingale case when their prices are equal to the prices of American calls; so if European call prices do decrease at some point, then the underlying risky process must be a strict local martingale under a risk neutral measure, whence one has a bubble. Conversely, these considerations can provide evidence (albeit not definitive evidence) that a bubble is not occurring, if European call prices behave appropriately even for long time horizons.

1 A complete analysis for the 3-dimensional Bessel process

Throughout this section we will refer as the *canonical space* the space of continuous function $C[0, \infty)$ together with the right-continuous filtration

obtained from the natural filtration of the coordinate process. The laws of all continuous stochastic processes are probability measures on this space.

Let X_t be a BES(3) process starting from a nonnegative constant x_0 . In other words (see [19, page 158] for the details), X is the (strong) solution of the stochastic differential equation:

$$dX_t = \frac{1}{X_t} dt + d\beta_t, \quad X_0 = x_0, \quad (1)$$

where β is an one-dimensional standard Brownian motion. One can think of X as the Euclidean norm process of a three dimensional standard Brownian motion \tilde{B} , issuing from b_0 satisfying $\|b_0\| = x_0$. If $x_0 > 0$, then without loss of generality (by resorting to scaling) we can assume that $x_0 = 1$. By the transience of the three-dimensional Brownian motion it follows that almost surely $1/X_t$ is well-defined for all $t \geq 0$. This reciprocal process $\{1/X_t, t \geq 0\}$ is probably the most well-known example of strict local martingale, in spite of the fact that the family of random variables is uniformly integrable on the entire range of t .

Proposition 2. *Let X_t be a BES(3) process starting from one. For any real $K \in [0, 1/2]$, the function $t \mapsto E\{(1/X_t - K)^+\}$ is strictly decreasing for all $t \in (0, \infty)$. However, if $K > 1/2$, the function $t \mapsto E\{(1/X_t - K)^+\}$ is initially increasing and then strictly decreasing for*

$$t \geq \left(K \log \frac{2K + 1}{2K - 1} \right)^{-1}.$$

Remark: Note that the bound on the right hand becomes zero when $K = 1/2$ which demonstrates its sharpness. Also, note the transition at $K = 1/2$, which is surprising since, at first, one would expect a transition at $K = 1$. However, it turns out that $1/2$ is in fact a result of the first Ray-Knight theorem describing local time of Brownian motion till it hits a level.

The proof uses a change of measure technique which transforms a Brownian motion into a BES(3) process as we state below.

Theorem *Let X_t denote the coordinate process on the canonical space, and let Q denote the Wiener measure such that $Q(X_0 = 1) = 1$. Let τ_0 denote the first hitting time of zero, i.e., $\tau_0 = \inf\{t \geq 0, X_t = 0\}$. Then $X_{t \wedge \tau_0}$ is a martingale under Q and $E(X_{t \wedge \tau_0}) = 1$. Define a measure P by the domination relation*

$$\left. \frac{dP}{dQ} \right|_{\mathcal{F}_t} = X_{t \wedge \tau_0}. \quad (2)$$

Then, under P , the law of the coordinate process X_t is BES(3) with $X_0 = 1$.

Proof. Use Girsanov's theorem to see that under P the coordinate process satisfies the relation

$$X_t = \beta_t + \int_0^t \frac{ds}{X_{s \wedge \tau_0}}.$$

It then follows that X must be strictly positive throughout and satisfy SDE (1). Since the function $f(x) = \frac{1}{x}$ is locally Lipschitz on $(0, \infty)$, we have the uniqueness of the solution of the SDE, and the theorem follows. \square

Proof of Proposition 2. From the change of measure relationship (2) we deduce the following identity

$$\begin{aligned} E^P \left\{ \left(\frac{1}{X_t} - K \right)^+ \right\} &= E^Q \left\{ X_{t \wedge \tau_0} \left(\frac{1}{X_t} - K \right)^+ \right\} \\ &= E^Q \left[X_t \left(\frac{1}{X_t} - K \right)^+ 1_{\{\tau_0 > t\}} \right]. \end{aligned}$$

If $x > 0$, then $x(1/x - K)^+ = (1 - Kx)^+$, and thus

$$E^P \left\{ \left(\frac{1}{X_t} - K \right)^+ \right\} = E^Q [(1 - KX_t)^+ 1_{\{\tau_0 > t\}}]. \quad (3)$$

It follows from the last equation that if X denotes a BES(3) process and B denotes a Brownian motion, both starting from one, we get

$$E \left\{ \left(\frac{1}{X_t} - K \right)^+ \right\} = E [(1 - KB_t)^+ 1_{\{\tau_0 > t\}}], \quad (4)$$

where τ_0 is the hitting time of zero for the Brownian motion B .

We will explicitly compute the right side of the above equation. Recall (see [19, page 97]) that for x, y , and t strictly positive the transition function of Brownian motion absorbed at zero is given by

$$p(t, x, y) := \frac{1}{\sqrt{2\pi t}} \left[\exp\left(-\frac{(y-x)^2}{2t}\right) - \exp\left(-\frac{(y+x)^2}{2t}\right) \right].$$

Thus, when the processes start at one,

$$\begin{aligned} E [(1 - KB_t)^+ 1_{\{\tau_0 > t\}}] &= \int_0^\infty (1 - Ky)^+ p(t, 1, y) dy \\ &= \frac{1}{\sqrt{2\pi t}} \int_0^\infty (1 - Ky)^+ \exp\left(-\frac{(y-1)^2}{2t}\right) dy \\ &\quad - \frac{1}{\sqrt{2\pi t}} \int_0^\infty (1 - Ky)^+ \exp\left(-\frac{(y+1)^2}{2t}\right) dy \\ &= h_1(t) - h_2(t), \quad (\text{say}). \end{aligned} \quad (5)$$

Now, the function h_1 is given by

$$\begin{aligned}
h_1(t) &= \frac{1}{\sqrt{2\pi t}} \int_0^\infty (1 - Ky)^+ \exp\left(-\frac{(y-1)^2}{2t}\right) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-t^{-1/2}}^\infty (1 - K - K\sqrt{tz})^+ \exp(-z^2/2) dz, \quad z = \frac{y-1}{\sqrt{t}}, \\
&= \frac{1}{\sqrt{2\pi}} \int_{-t^{-1/2}}^{K_1 t^{-1/2}} (1 - K - K\sqrt{tz}) e^{-z^2/2} dz, \quad K_1 = \frac{1-K}{K}, \\
&= (1-K) \int_{-t^{-1/2}}^{K_1 t^{-1/2}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz - \frac{K\sqrt{t}}{\sqrt{2\pi}} \int_{-t^{-1/2}}^{K_1 t^{-1/2}} z e^{-z^2/2} dz \\
&= (1-K) \left[\Phi(K_1 t^{-1/2}) - \Phi(-t^{-1/2}) \right] + \frac{K\sqrt{t}}{\sqrt{2\pi}} \left(e^{-K_1^2/2t} - e^{-1/2t} \right).
\end{aligned}$$

Here Φ is the standard Gaussian distribution function.

Similarly, the second function h_2 in (5) is given by

$$\begin{aligned}
h_2(t) &= \frac{1}{\sqrt{2\pi t}} \int_0^\infty (1 - Ky)^+ \exp\left(-\frac{(y+1)^2}{2t}\right) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{t^{-1/2}}^\infty (1 + K - K\sqrt{tz})^+ e^{-z^2/2} dz, \quad z = \frac{y+1}{\sqrt{t}}, \\
&= \frac{1}{\sqrt{2\pi}} \int_{t^{-1/2}}^{K_2 t^{-1/2}} (1 + K - K\sqrt{tz}) e^{-z^2/2} dz, \quad K_2 = \frac{1+K}{K} \\
&= (1+K) \left[\Phi(K_2 t^{-1/2}) - \Phi(t^{-1/2}) \right] + \frac{K\sqrt{t}}{\sqrt{2\pi}} \left(e^{-K_2^2/2t} - e^{-1/2t} \right).
\end{aligned}$$

Thus, combining (4) and (5), we get

$$\begin{aligned}
h(t) &:= E\left(\frac{1}{X_t} - K\right)^+ = h_1(t) - h_2(t) \\
&= (1-K) \left[\Phi(K_1 t^{-1/2}) - \Phi(-t^{-1/2}) \right] + \frac{K\sqrt{t}}{\sqrt{2\pi}} e^{-K_1^2/2t} \\
&\quad - \frac{K\sqrt{t}}{\sqrt{2\pi}} e^{-K_2^2/2t} - (1+K) \left[\Phi(K_2 t^{-1/2}) - \Phi(t^{-1/2}) \right],
\end{aligned} \tag{6}$$

where the two constants K_1 and K_2 are given by

$$K_1 = \frac{1-K}{K}, \quad K_2 = \frac{1+K}{K}.$$

We need to show monotonicity properties of h as t increases. However, it is not easy to work with the derivatives of h . Hence we will make a change of variable by defining

$$x = 1/\sqrt{t}, \quad t \in (0, \infty), \quad \text{and} \quad g(x) = h(t) = h(1/x^2). \quad (7)$$

Showing h is monotonically decreasing (increasing) as t is equivalent to showing that g is monotonically increasing (decreasing) in x . This we achieve by differentiating g with respect to x . By (6)

$$\begin{aligned} g(x) = h(1/x^2) &= (1 - K)\Phi\left(\frac{(1 - K)x}{K}\right) - (1 - K)\Phi(-x) \\ &\quad + \frac{K}{x\sqrt{2\pi}} \exp\left[-\frac{(1 - K)^2 x^2}{2K^2}\right] - \frac{K}{x\sqrt{2\pi}} \exp\left[-\frac{(1 + K)^2 x^2}{2K^2}\right] \\ &\quad - (1 + K)\Phi\left(\frac{(1 + K)x}{K}\right) + (1 + K)\Phi(x). \end{aligned} \quad (8)$$

It is straightforward to differentiate the function above, and we get

$$\begin{aligned} g'(x) &= \frac{(1 - K)^2}{K\sqrt{2\pi}} \exp\left[-\frac{(1 - K)^2 x^2}{2K^2}\right] + \frac{(1 - K)}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right] \\ &\quad - \frac{K}{x^2\sqrt{2\pi}} \exp\left[-\frac{(1 - K)^2 x^2}{2K^2}\right] - \frac{(1 - K)^2}{K\sqrt{2\pi}} \exp\left[-\frac{(1 - K)^2 x^2}{2K^2}\right] \\ &\quad + \frac{K}{x^2\sqrt{2\pi}} \exp\left[-\frac{(1 + K)^2 x^2}{2K^2}\right] + \frac{(1 + K)^2}{K\sqrt{2\pi}} \exp\left[-\frac{(1 + K)^2 x^2}{2K^2}\right] \\ &\quad - \frac{(1 + K)^2}{K\sqrt{2\pi}} \exp\left[-\frac{(1 + K)^2 x^2}{2K^2}\right] + \frac{(1 + K)}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right]. \end{aligned}$$

Note that two pairs of terms (the first with the fourth and the sixth with the seventh) cancel, and we are left with

$$\begin{aligned} K\sqrt{2\pi}g'(x) &= K(1 - K) \exp\left(-\frac{x^2}{2}\right) - \frac{K^2}{x^2} \exp\left[-\frac{(1 - K)^2 x^2}{2K^2}\right] \\ &\quad + \frac{K^2}{x^2} \exp\left[-\frac{(1 + K)^2 x^2}{2K^2}\right] + K(1 + K) \exp\left[-\frac{x^2}{2}\right] \\ &= 2K \exp\left(-\frac{x^2}{2}\right) - \frac{K^2}{x^2} \left\{ \exp\left[-\frac{(1 - K)^2 x^2}{2K^2}\right] - \exp\left[-\frac{(1 + K)^2 x^2}{2K^2}\right] \right\}. \end{aligned}$$

Thus $g'(x) > 0$ if and only if

$$\begin{aligned}
2K \exp\left(-\frac{x^2}{2}\right) &> \frac{K^2}{x^2} \left\{ \exp\left[-\frac{(1-K)^2 x^2}{2K^2}\right] - \exp\left[-\frac{(1+K)^2 x^2}{2K^2}\right] \right\} \\
\Leftrightarrow 2x^2 &> K e^{x^2/2} \left\{ \exp\left[-\frac{(1-K)^2 x^2}{2K^2}\right] - \exp\left[-\frac{(1+K)^2 x^2}{2K^2}\right] \right\} \\
\Leftrightarrow 2x^2 &> K \left\{ \exp\left[\frac{(2K-1)x^2}{2K^2}\right] - \exp\left[-\frac{(2K+1)x^2}{2K^2}\right] \right\}.
\end{aligned} \tag{9}$$

We need to do a bit more work. Consider the function on the right side of the last inequality

$$\exp\left[\frac{(2K-1)x^2}{2K^2}\right] - \exp\left[-\frac{(2K+1)x^2}{2K^2}\right].$$

We need to consider two separate cases. First suppose $K > 1/2$. Then both $2K-1$ and $2K+1$ are positive. If for two positive parameters $\lambda_2 > \lambda_1 > 0$, we define a function q by $q(y) = \exp(\lambda_1 y) - \exp(-\lambda_2 y)$, $y > 0$, it then follows that

$$\begin{aligned}
q'(y) &= \lambda_1 e^{\lambda_1 y} + \lambda_2 e^{-\lambda_2 y}, \quad q'(0) = \lambda_1 + \lambda_2, \\
q''(y) &= \lambda_1^2 e^{\lambda_1 y} - \lambda_2^2 e^{-\lambda_2 y}.
\end{aligned} \tag{10}$$

Note that $q''(y) < 0$, for all

$$0 \leq y < \frac{2 \log(\lambda_2/\lambda_1)}{\lambda_1 + \lambda_2}. \tag{11}$$

Since $q'(y)$ is always positive, it follows that q is an increasing concave function starting from zero in the interval given by (11). Thus it also follows that, in that interval,

$$q(y) = q(y) - q(0) < yq'(0) = y(\lambda_1 + \lambda_2). \tag{12}$$

Take $y = x^2$, $\lambda_1 = (2K-1)/2K^2$ and $\lambda_2 = (2K+1)/2K^2$. Then $\lambda_1 + \lambda_2 = 2/K$. By (11) we get that if

$$x \leq C_1 := \sqrt{K \log \frac{2K+1}{2K-1}},$$

then, from (12) it follows

$$K \left\{ \exp\left[\frac{(2K-1)x^2}{2K^2}\right] - \exp\left[-\frac{(2K+1)x^2}{2K^2}\right] \right\} < 2x^2.$$

That is, by (9), $g'(x) > 0$ for all $x \in [0, C_1]$. Thus, g is increasing in $[0, C_1]$. Changing our variable back to $t = 1/x^2$, we get that $h(t)$ is strictly decreasing for all

$$t > \left(K \log \frac{2K+1}{2K-1} \right)^{-1}.$$

The case when $0 < K \leq 1/2$ can be handled similarly. Suppose $0 < \lambda_1 < \lambda_2$ are positive constants. Consider the function

$$r(y) = -\lambda_1 y + \lambda_2 y - e^{-\lambda_1 y} + e^{-\lambda_2 y}, \quad y \in [0, \infty).$$

Then $r(0) = 0$, and

$$r'(y) = -\lambda_1 (1 - e^{-\lambda_1 y}) + \lambda_2 (1 - e^{-\lambda_2 y}) > 0, \quad y \in [0, \infty),$$

because $\lambda_1 < \lambda_2$. Thus, for all positive y , we have $r(y) > 0$, i.e.,

$$e^{-\lambda_1 y} - e^{-\lambda_2 y} < (-\lambda_1 + \lambda_2)y.$$

We use this for $\lambda_1 = (1 - 2K)/2K^2$ and $\lambda_2 = (1 + 2K)/2K^2$, and $y = x^2$. Note that, as before

$$(-\lambda_1 + \lambda_2)x^2 = 2x^2/K.$$

From (9) it follows that $g'(x) > 0$ for all $x \in (0, \infty)$. Reverting back to t , we establish that if $K \leq 1/2$, the function $t \mapsto E(1/X_t - K)^+$ is strictly decreasing for all $t \in (0, \infty)$. This completes the proof of the proposition. \square

For mathematical completeness we show below that a similar result can be proved for the Bessel process starting from zero, although in this case there is no dependence on K . The proof is much simpler and essentially follows by a scaling argument. Note that, even in this case the reciprocal of the Bessel process is well-defined for all times except at time zero. Hence $1/X_t$, $t \in (0, \infty)$, can be thought as a Markov process with an *entrance distribution*, i.e., a pair consisting of a time-homogenous Markov transition kernel $\{P_t\}$, $t > 0$, and a family of probability measures $\{\mu_s\}$, $s > 0$, satisfying the constraint $\mu_s * P_t = P_{t+s}$. Here $*$ refers to the action of the kernel on the measure.

Proposition 3. *Let X_t be a 3-dimensional Bessel process, $BES(3)$, such that $X_0 = 0$. For any two time points $u > t > 0$, and for $K \geq 0$, one has*

$$E \left(\frac{1}{X_u} - K \right)^+ < E \left(\frac{1}{X_t} - K \right)^+. \quad (13)$$

Proof. Fix $u > t$. Recall that BES(3), being the norm of a three dimensional Brownian motion, has the Brownian scaling property when starting from zero. That is to say, for any $c > 0$,

$$\left(\frac{1}{\sqrt{c}} X_{cs}, s \geq 0 \right) \stackrel{\mathcal{L}}{=} (X_s, s \geq 0),$$

where the above equality is equality in law.

Take $c = u/t$, and apply the above equality for X_s when $s = t$, to infer that $c^{-1/2}X_u$ has the same law as X_t , and thus

$$E \left(\frac{1}{X_u} - K \right)^+ = E \left(\frac{c^{-1/2}}{X_t} - K \right)^+ = c^{-1/2} E \left(\frac{1}{X_t} - \sqrt{c}K \right)^+. \quad (14)$$

Note that for any $\sigma > 1$, we have

$$\frac{1}{\sigma} (x - \sigma K)^+ < (x - K)^+, \quad \forall x > 0.$$

Since $c > 1$, taking $\sigma = \sqrt{c}$, one deduces from (14)

$$E \left(\frac{1}{X_u} - K \right)^+ < E \left(\frac{1}{X_t} - K \right)^+,$$

which proves the result. □

2 A more general result

How do the results in the last section apply to the general class of continuous, strictly positive, strict local martingales ? As we shall see, the answer is delicate and there seems to be no general uniformity in behavior even among this class.

Let $\{N_t, t \geq 0\}$ be an arbitrary positive strict local martingale with $N_0 = 1$. We first prove Proposition 1 claimed in the introduction.

Proof of Proposition 1. Since N is a nonnegative local martingale, it is also a supermartingale. The proposition now follows easily from the observation that for any $K > 0$, by the nonnegativity of the process N , we have $(N_t - K)^+ \leq N_t$, and hence

$$E(N_t - K)^+ \leq EN_t.$$

By the supermartingale property, $E(N_t)$ is monotonically decreasing in t , and decreasing to zero by our assumption. Hence, for any u , if t_0 is the first time t such that $E(N_{t_0}) < E\{(N_u - K)^+\}$, then for all $t \geq t_0$, we have

$$E\{(N_t - K)^+\} \leq E(N_t) \leq E(N_{t_0}) < E\{(N_u - K)^+\}.$$

This proves the proposition. \square

Proposition 1 shows that if $\lim_{t \rightarrow \infty} E(N_t) = 0$, the curve $t \mapsto E(N_t - K)^+$, which is always dominated by the curve $t \mapsto E(N_t)$, is converging to zero. However, this does not show the monotonicity properties of $E\{(N_t - K)^+\}$ itself. A finer analysis requires the following lemma which generalizes Doob's h -transform result for the BES(3) process which we used so crucially in the last section. As we mentioned earlier, this was proved by Delbaen and Schachermayer [5] in 1995. We replicate their theorem below. This construction is related to the Föllmer measure of a positive supermartingale [9]. The technique of h -transform itself has also been applied to several other problems, for example the analysis of Brownian meander by Biane and Yor [1]. Before we state the result we need a technique which adds an extra absorbing point to the state space \mathbb{R}^+ . Our treatment is inspired by the work of P. A. Meyer [23], and we follow the notation used in [5] closely.

The space of trajectories is the space $C_\infty[0, T]$ or $C_\infty[0, \infty)$ of continuous paths ω defined on the time interval $[0, T]$ or $[0, \infty)$ with values in $[0, \infty]$ with the extra property that if $\omega(t) = \infty$, then $\omega(s) = \infty$ for all $s > t$. The topology endowed is the one associated with local uniform convergence. The coordinate process is denoted by L , i.e., $L(t) = \omega(t)$. One needs a somewhat careful dealing with the filtration which we skip completely and request the reader to look into [5] for the details.

Theorem 1 (Delbaen and Schachermayer, [5] Theorem 4). *If R is a measure on $C[0, 1]$ such that L is a strictly positive strict local martingale, then*

- (i) *there is a probability measure R^* on $C_\infty[0, 1]$ such that $M = 1/L$ is an R^* martingale.*
- (ii) *We may choose R^* in such a way that the measure R is absolutely continuous with respect to R^* and its Radon-Nikodým derivative is given by $dR = M_1 dR^*$.*

The following result is an immediate corollary.

Lemma 4. *Let N_t , $t \geq 0$, be a continuous strictly positive local martingale such that $N_0 = 1$. Then there exists a nonnegative martingale M on the canonical space such that the following holds.*

(i) The probability measure defined by

$$P(A) := E(M_t 1_A), \quad \forall A \in \mathcal{F}_t, t \geq 0, \quad (15)$$

is the law of the process $\{1/N_t, t \geq 0\}$.

(ii) N is a strict local martingale if and only if $Q(\tau_0 < \infty) > 0$, where $\tau_0 = \inf\{t \geq 0 : M_t = 0\}$ and Q is the law of M . Moreover, we have $E(N_t) = P(\tau_0 > t)$ for all $t \in (0, \infty)$.

Conversely, let $\{M_t, t \geq 0\}$ be a nonnegative martingale on the canonical space starting from one. Let Q denote the law of M , and assume that $Q(\tau_0 < \infty) > 0$. Consider the change of measure

$$P(A) := E(M_{t \wedge \tau_0} 1_A), \quad \forall A \in \mathcal{F}_t, t \geq 0.$$

Then, under P , the process $1/M_t$ is a positive strict local martingale.

Proof. The only difference in the first part of this lemma with the previous Theorem is that the construction is on the entire space $C[0, \infty)$. Consider the law of the local martingale L on the canonical space. Note that, by scaling time, Theorem 1 holds for any time interval $[0, T]$, $T = 1, 2, \dots$. In other words, for every positive integer T , there is a martingale $M_t(T)$, $0 \leq t \leq T$, which satisfies the two conditions in Theorem 1 in time interval $[0, T]$. Let Q_T be the law of $M(T)$ on the σ -algebra \mathcal{F}_T generated by the coordinate process up to time T . Once we demonstrate that this tower of probability measures is consistent, it follows from standard arguments that they induce a probability measure Q on the entire space $C_\infty[0, \infty)$ with the required properties holding locally. However, consistency is immediate once both the properties (i) and (ii) in Theorem 1 hold for each interval $[0, T]$.

Part (ii) follows once we show that $E(N_\tau) = Q(\tau_0 > \tau)$ for all bounded stopping times τ . Indeed, if $Q(\tau_0 = \infty) = 1$, the claim shows that $E(N_\tau) = 1$ for all bounded stopping times τ establishing its martingale property. To show $E(N_\tau) = Q(\tau_0 > \tau)$, note that, since τ is bounded, the absolute continuity relation (15) holds on \mathcal{F}_τ , and we immediately derive

$$E(N_\tau) = E\left(\frac{1}{M_\tau} M_\tau 1_{\{\tau_0 > \tau\}}\right) = Q(\tau_0 > \tau).$$

The converse is easier to show. A detailed account can be found in the proof of Theorem 1 in [5]. In short, one shows $1/M$ is a local martingale under P by using Girsanov's theorem. Since it is nonnegative, it must hence be a supermartingale. That it is not a martingale follows from what we just showed: $E(N_t) = Q(\tau_0 > t)$. \square

Other than the example of BES(3) and the Brownian motion that we have encountered, several other examples of Lemma 4 can be derived. In particular, for any Bessel process X of dimension $\delta > 2$, it is well-known that $X^{2-\delta}$ is a strict local martingale. The law of its reciprocal can be derived in a procedure similar to Lemma 4 (see [21]) using the martingale $Y^{\delta-2}$, where Y is a Bessel process of (possibly negative) dimension $(4-\delta)$. Thus inspired by the previous representation theorem, we make the following definition:

Definition 1. *We will call an ordered pair of continuous processes (N, M) to be a Girsanov pair if*

1. $\{N_t, t \geq 0\}$ is a positive strict local martingale starting from one.
2. $\{M_t, t \geq 0\}$ is a nonnegative martingale.
3. The laws of M and N are related by Lemma 4. In particular, for any bounded function $F : (0, \infty) \rightarrow \mathbb{R}$, one has

$$E(F(N_t)) = E\left[M_t F\left(\frac{1}{M_t}\right) 1_{\{\tau_0 > t\}}\right],$$

where $\tau_0 = \{t \geq 0, M_t = 0\}$ is the first hitting time of zero for M .

If (N, M) is a Girsanov pair according to this definition, it follows that, for any $K > 0$, analyzing the function $t \mapsto E\{(N_t - K)^+\}$ is the same as analyzing the function

$$t \mapsto E\left[M_t 1_{\{\tau_0 > t\}} \left(\frac{1}{M_t} - K\right)^+\right] = E[(1 - KM_t)^+ 1_{\{\tau_0 > t\}}].$$

Now for any $K > 0$ and any nonnegative path M which gets absorbed upon hitting zero, the following is an algebraic identity:

$$(1 - KM_t)^+ 1_{\{\tau_0 > t\}} = (1 - KM_t)^+ - 1_{\{\tau_0 \leq t\}}.$$

In particular this identity holds pathwise when M is a nonnegative martingale. Taking expectation on both sides of the last equation we obtain

$$E[(1 - KM_t)^+ 1_{\{\tau_0 > t\}}] = E(1 - KM_t)^+ - P(\tau_0 \leq t).$$

Using Tanaka's formula applied to the martingale M , we get the following semimartingale decomposition

$$(1 - KM_t)^+ = (1 - KM_0)^+ - K \int_0^t 1_{\{M_s < 1/K\}} dM_s + \frac{K}{2} L_t^{1/K}(M). \quad (16)$$

The process $L^{1/K}(M)$ is the local time at $1/K$ of the process M . Taking expectation on both sides of (16), we obtain

$$E(1 - KM_t)^+ = (1 - KM_0)^+ + \frac{K}{2}E\left(L_t^{1/K}(M)\right), \quad t \geq 0.$$

Combining these observations we obtain the next proposition which is true for all positive, continuous, strict local martingales. We start with the following definition.

Definition 2. *We call a mapping $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ to be asymptotically decreasing if there exists a $t_0 > 0$, such that for all $t \geq t_0$, the curve $t \mapsto h(t)$ is decreasing.*

Proposition 5. *Let $(N_t, t \geq 0)$ be a positive, continuous, strict local martingale such that $N_0 = 1$. Let M be a nonnegative martingale such that (N, M) is a Girsanov pair. Then, for any $K > 0$, the following are equivalent.*

- (i) *The map $t \mapsto E\{(N_t - K)^+\}$ is asymptotically decreasing.*
- (ii) *The map $t \mapsto E[(1 - KM_t)^+ 1_{\{\tau_0 > t\}}]$ is asymptotically decreasing, where $\tau_0 = \inf\{t \geq 0 : M_t = 0\}$.*
- (iii) *The map $t \mapsto E\{(1 - KM_t)^+\} - P(\tau_0 \leq t)$ is asymptotically decreasing.*
- (iv) *The map*

$$t \mapsto \frac{K}{2}E\left(L_t^{1/K}(M)\right) - P(\tau_0 \leq t)$$

is asymptotically decreasing, where the process $L^{1/K}(M)$ is the local time at $1/K$ of the martingale M .

- (v) *The map*

$$t \mapsto \frac{1}{2}E\left(L_t^K(N)\right) + E(N_t)$$

is asymptotically decreasing, where the process $L^K(N)$ is the local time at K of the local martingale N .

Proof. In light of our previous discussion, we only need to show the equivalence of conditions (iv) and (v). Since we have already shown $E(N_t) = P(\tau_0 > t) = 1 - P(\tau_0 \leq t)$, it suffices to prove

$$KE\left(L_t^{1/K}(M)\right) = E\left(L_t^K(N)\right). \quad (17)$$

To prove this we use the *coupling* of M and $N = 1/M$ as obtained from Lemma 4. Note that for any measurable compactly supported function $f : (0, \infty) \rightarrow (0, \infty)$ we can use the occupation time formula involving the local time for general continuous semimartingales (see, for example, [19, page 218], although our definition of local time differs from [19] by a factor of 2) to obtain

$$\int_0^\infty f(a) L_t^a(N) da = \int_0^t f(N_s) d\langle N_s \rangle = \int_0^t f\left(\frac{1}{M_s}\right) \frac{1}{M_s^4} d\langle M_s \rangle$$

By the change of measure formula (15), we obtain

$$\begin{aligned} E \left[\int_0^\infty f(a) L_t^a(N) da \right] &= E \left[M_t \int_0^t f\left(\frac{1}{M_s}\right) \frac{1}{M_s^4} d\langle M_s \rangle \right] \\ &= E \left[\int_0^{t \wedge \tau_0} M_s f\left(\frac{1}{M_s}\right) \frac{1}{M_s^4} d\langle M_s \rangle \right]. \end{aligned} \quad (18)$$

The final equality is due to the integration by parts formula and the fact that M is a martingale. Using now the occupation time formula for the martingale M , we get

$$\begin{aligned} E \left[\int_0^{t \wedge \tau_0} f\left(\frac{1}{M_s}\right) \frac{1}{M_s^3} d\langle M_s \rangle \right] &= E \int_0^\infty f\left(\frac{1}{b}\right) L_t^b(M) \frac{db}{b^3} \\ &= E \int_0^\infty f(a) a L_t^{1/a}(M) da, \quad a = 1/b. \end{aligned} \quad (19)$$

Combining (18) and (19) and using Fubini-Tonelli for nonnegative integrands to interchange expectations and integrals, we obtain

$$\int_0^\infty f(a) E(L_t^a(N)) da = \int_0^\infty f(a) a E(L_t^{1/a}(M)) da.$$

Since this holds for all nonnegative functions with compact support in $(0, \infty)$, we have proved (17).

Alternatively, this equivalence follows from Theorem 1 in Madan and Yor [21]. To prove Theorem 1, the authors use machinery which is very different from h -transforms. However, the method shown in the previous paragraph essentially furnishes another proof of the same result. \square

Unfortunately none of these equivalent conditions is simple to check in general. Notice, however, that to show the final equivalent condition (iv) or

(v) in the last proposition, it suffices to prove

$$\begin{aligned} & \frac{K}{2} \frac{d}{dt} E \left(L_t^{1/K}(M) \right) - p_t(0) \leq 0, \quad \text{for all large enough } t, \\ \text{or, } & \frac{1}{2} \frac{d}{dt} E \left(L_t^{1/K}(N) \right) - \frac{d}{dt} E(N_t) \leq 0, \quad \text{for all large enough } t. \end{aligned} \quad (20)$$

where

$$p_t(0) = \frac{d}{dt} P(\tau_0 \leq t) \quad (21)$$

is the density of the first hitting time of zero for the martingale M . In the next proposition we show that one can simplify the derivative of the expected local time under a Markovian assumption on either of the processes.

Lemma 6. *Suppose $\{X_t, t \geq 0\}$ is a continuous nonnegative local martingale which satisfies the following SDE*

$$dX_t = \sigma(t, X_t) d\beta_t, \quad t \in [0, \infty), \quad X_0 = 1. \quad (22)$$

Here β is a one-dimensional standard Brownian motion and $\sigma(t, x)$ is some measurable nonnegative function on $\mathbb{R}^+ \times \mathbb{R}^+$.

Further assume that the process X_t admits a continuous marginal density at each time t at every strict positive point y which is given by

$$p_t(y) = P \left(X_t \in dy \mid X_0 = 1 \right), \quad y > 0.$$

Let L_t^a denote the local time of X at level $a > 0$ and at time t . Then

$$\frac{d}{dt} E(L_t^a) = \sigma^2(t, a) p_t(a). \quad (23)$$

Proof. To prove this, we again use the occupation time formula involving the local time for general continuous semimartingales.

For any smooth nonnegative function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ with compact support contained in $(0, \infty)$, we have the following identity

$$\int_{\mathbb{R}^+} f(a) L_t^a da = \int_0^t f(X_s) d\langle X \rangle_s = \int_0^t f(X_s) \sigma^2(s, X_s) ds,$$

where the final identity follows from (22). Now taking expectations on both sides, we obtain

$$\begin{aligned} E \left[\int_{\mathbb{R}^+} f(a) L_t^a da \right] &= E \int_0^t f(X_s) \sigma^2(s, X_s) ds = \int_0^t E [f(X_s) \sigma^2(s, X_s)] ds \\ &= \int_0^t \left[\int_{\mathbb{R}^+} f(a) \sigma^2(s, a) p_s(a) da \right] ds. \end{aligned} \quad (24)$$

The second equality above is due to Fubini-Tonelli for nonnegative integrands. The final equality is by definition of the marginal density and the fact that the support of f is in $(0, \infty)$.

Applying Fubini-Tonelli repeatedly and interchanging the orders of integration on both sides of (24), we get

$$\begin{aligned} \int_{\mathbb{R}^+} f(a)E(L_t^a)da &= E \left[\int_{\mathbb{R}^+} f(a)L_t^a da \right] = \int_0^t \left[\int_{\mathbb{R}^+} f(a)\sigma^2(s, a)p_s(a)da \right] ds \\ &= \int_{\mathbb{R}^+} f(a) \left[\int_0^t \sigma^2(s, a)p_s(a)ds \right] da. \end{aligned}$$

Since this holds for all smooth nonnegative functions f with compact support in $(0, \infty)$, it follows that

$$E(L_t^a) = \int_0^t \sigma^2(s, a)p_s(a)ds, \quad \forall a > 0.$$

The conclusion of the lemma is now obvious. \square

The advantage of the following (and our final) proposition is that the criterion of asymptotic monotonic decay of call price depends on the transition densities of the stock price process, whose bounds or decay rate might be easier to compute from modeling perspectives.

Proposition 7. *Suppose (N, M) is a Girsanov pair such that $N_0 = M_0 = 1$. Assume that N (or M) satisfies the hypotheses of Lemma 6 with volatility σ (or ρ). Then, for some $K > 0$, the map $t \mapsto E\{(N_t - K)^+\}$ is asymptotically decreasing if and only if*

$$\begin{aligned} \limsup_{t \rightarrow \infty} \left[\sigma^2(t, K)p_t(K) + 2\frac{d}{dt}E(N_t) \right] &< 0, \quad \text{or,} \\ \limsup_{t \rightarrow \infty} [K\rho^2(t, 1/K)q_t(1/K) - 2q_t(0)] &< 0, \end{aligned} \tag{25}$$

where $p_t(K)$ is the density of N_t at K , $q_t(1/K)$ is the density of M_t at $1/K$, and $q_t(0)$ is defined as in (21).

Proof. The proof follows from the equivalence of conditions (i) and (iv) in Proposition 5 by applying (20) and Lemma 6. \square

3 A counterexample to the phenomenon

We finally demonstrate that the asymptotic decreasing property of the function $E(N_t - K)^+$ is hardly universal among the class of positive strict local martingales. In this section we demonstrate an example of a strict local martingale N where the asymptotically decreasing property of $E\{(N_t - K)^+\}$ does not hold for any positive K . The way to construct N is through its Girsanov pair M and to use Lemma 4.

We construct a nonnegative martingale M adaptively on each time interval $[i, i + 1]$, for $i = 0, 1, 2, \dots$, starting with $M_0 = 1$. Like all nonnegative martingales, it gets absorbed at any time it hits zero. In the interval $[0, 1]$, M is a Brownian motion starting from one and getting absorbed upon hitting zero. In case $M_1 > 0$, the process M is defined in the interval $[1, 2]$ to be a Brownian motion starting from M_1 and absorbed the first time it hits $M_1/2$. We repeat this procedure in the following way. In every interval of the type $[2i, 2i + 1]$, $i \in \mathbb{N}$, M is a Brownian motion starting from M_{2i} and getting absorbed upon hitting zero. If $M_{2i+1} > 0$, during the next time interval $[2i + 1, 2i + 2]$, the process is a Brownian motion starting from M_{2i+1} and getting absorbed if it hits $M_{2i+1}/2$. If at any point of time it hits zero, it gets absorbed forever in the future.

To see that M is a nonnegative martingale, notice that in every interval, the increment of the process is a Brownian motion stopped by a bounded stopping time. Also the bracket process $\langle M \rangle$ increase linearly on every even interval until the process hits zero. Thus

$$\langle M \rangle_{2i} \geq i \wedge \tau_0, \quad i \in \mathbb{N}.$$

By the Dambis-Dubins-Schwartz theorem ([19, page 174]), it follows that $P(\tau_0 < \infty) = 1$. Thus, by Lemma 4, there exists a strictly positive strict local martingale N such that (N, M) is a Girsanov pair. In particular, $N_0 = 1$ and $\lim_{t \rightarrow \infty} E(N_t) = 0$.

Next for any $K > 0$, consider the map $t \mapsto E(N_t - K)^+$. To analyze the monotonicity properties of this map we shall utilize the equivalence of (i) and (iv) in Proposition 5. Fix any $K > 0$. It follows from the construction of M that during any time interval $t \in [2i + 1, 2i + 2]$, $i \in \mathbb{N}$, the probability $P(\tau_0 \leq t)$ does not change. However, there is a positive chance of the process M hitting $1/K$ during that same time interval. Hence $E(L_t^{1/K})$ is increasing. Thus, in every such time interval $t \in [2i + 1, 2i + 2]$, the quantity

$$\frac{1}{2} E \left(L_t^{1/K} \right) - P(\tau_0 \leq t)$$

is increasing. By the equivalence in Proposition 5, in all such intervals the map $t \mapsto E(N_t - K)^+$ is actually increasing. Thus N is a strict local martingale for which the $E(N_t - K)^+$ is not asymptotically decreasing for any positive $K > 0$.

One might object to the fact that we are preventing M from hitting zero on odd intervals. But, one can easily construct a more complicated structure in which at every odd interval $[2i + 1, 2i + 2]$, we toss a coin with probability of head $p(M_{2i+1})$ such that

- (i) if Heads comes up, M is a Brownian motion starting at M_{2i+1} getting absorbed at $M_{2i+1}/2$.
- (ii) But if Tails comes up, M is a Brownian motion starting at M_{2i+1} and absorbed upon hitting zero.

M will retain its martingale property under such a *mixture*. Further, it is intuitively clear (although clumsy to write down exact probabilities) that if M_{2i+1} is not more than $2/K$, one can choose $p(M_{2i+1})$ carefully so as to still have $E\left(L_t^{1/K}\right) - 2P(\tau_0 \leq t)$ increasing. Since M_{2i+1} always has a positive probability of going above $2/K$, we have constructed a martingale which can hit zero at any time during $(0, \infty)$, but still leads to a strict local martingale with desired properties.

Acknowledgements. We are grateful to Professors Marc Yor and Monique Jeanblanc who drew our attention to the fascinating papers by Madan and Yor [21] and Elworthy, Li and Yor [8]. The second author gratefully acknowledges benefitting from a Fulbright-Tocqueville Distinguished Chair award at the University of Paris – Dauphine, during the development of this research.

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