Logic and Computation in Finitely Presentable Infinite Structures

Lecture 10: Model Checking Expressive Modal Logic on Infinite Structures

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ESSLLI 2006, Malaga, August 2006
Extended Modal Logics on Infinite Structures

The basic modal language is too weak for most practical applications of infinite systems. It can only express local properties. More expressive languages are needed.

Remark: All modal logics discussed here are subsumed by MSO, and hence decidable on e.g., all prefix-recognizable graphs. However, this class is restricted and the general complexity of MSO on prefix-recognizable graphs is non-elementary.
Some important directions and recent developments

- Modal logic with reachability.
- [Demri, Finkel, G., van Drimmelen, 2005]: decidable model checking of first-order CTL* over a class of Presburger-definable transition systems.
- [Löding, 2002]: decidable model checking of a fragment of CTL* (with recurrent, but not universal reachability) on regular ground term rewriting systems.
- [Walukiewicz, 2000]: EXPTIME model checking CTL on pushdown systems.
- [Kupferman, Piterman, and Vardi, 2002]: EXPTIME model checking LTL on prefix-recognizable systems.
Reachability

Given a labeled transition system $G = (W, \{R_a\}_{a \in A})$, a set of initial states $I$ and a set of target states or bad states $F$:

- **Reachability**: Is any state in $F$ reachable by a path in $G$ starting from a state in $I$?
- **Recurrent reachability**: Is the set $F$ reachable infinitely often by a path in $G$ starting from a state in $I$?
- **Universal (recurrent) reachability**: Does every path in $G$ starting from $I$ reach (infinitely often) a state in $I$?

**Reachability queries**:

- **Forward reachability**: Given $I$ compute the set of $\text{Reach}(I)$ of all states in $G$ reachable by a path from a state in $I$.
- **Backward reachability**: Given $F$ compute the set of $\text{Reach}^{-1}(F)$ of all states in $G$ from which a state in $F$ is reachable.
Modal Logic with Reachability

Extend the basic modal logic over a transition relation $R$ with a modality $[R^*]$ over the transitive closure of $R$; possibly, add $[R^{-*}]$ for the transitive closure of the inverse relation.

Then, some reachability queries are definable:

- $\langle R^* \rangle X = \{ x \mid R^*(x) \cap X \neq \emptyset \}$.
- $\langle R^{-*} \rangle X = \{ x \mid R^{-*}(x) \cap X \neq \emptyset \}$.

For repeated and universal reachability we need path quantifiers:

$$\forall \langle R^* \rangle X, \exists [R^*] \langle R^* \rangle X,$$

etc.
Some cases of decidable reachability in rational Kripke models

- Length-preserving and length-monotone rational relations.
- Models with finite bisimulation index. If a finite bisimulation exists for a given rational model, it can be computed effectively. Then, model-checking of the whole $\mu$-calculus is decidable in that model.
  Existence of a finite bisimulation is semi-decidable. Question: is existence of a finite bisimulation decidable?
  In case of pushdown graphs: yes (Stirling, 2000).
- More generally: consider rational models bisimular with prefix recognizable graphs, etc., all the way up Cauca hierarchy.
A new hierarchy

The modal mu-calculus is invariant under bisimulations and is embedded into MSO. Bisimulations do not preserve MSO. These suggest building a hierarchy of infinite Kripke models with decidable \( \mu \)-calculus, by starting with finite graphs and alternating bisimulations and MSO interpretations.

This hierarchy subsumes the pushdown hierarchy and extends it properly already on level 1; e.g. the grid \( \mathbb{N} \times \mathbb{N} \) appears there.

Questions: Is this hierarchy strict? Any explicit characterization of the lowest levels?
Presburger transition systems

Presburger control graph of dimension $n$: a finite digraph with edges $(i, j)$ labelled by Presburger formulae $\Psi_{ij}(\bar{x}, \bar{y})$, where $\bar{x}$ and $\bar{y}$ are disjoint $n$-tuples of variables.

Presburger transition system (PTS) with $n$ counters: $\mathbf{P} = \langle C, \mathcal{C} \rangle$, where $C$ is a Presburger control graph of dimension $n$ and $\mathcal{C} \subseteq Q \times \mathbb{N}^n$ is a set of configurations.

The transition relation $R_\Psi \subseteq C \times C$ in $\mathbf{P}$ is defined by the control formulae $\Psi_{ij}$, i.e.:

$$(i, \bar{s}) R_\Psi (j, \bar{t}) \text{ holds iff } \mathbb{N} \models \Psi_{ij}(\bar{s}, \bar{t}).$$

$\mathbf{P}$ has Presburger definable reachability if there is a Presburger formula $\Psi^*_P(\bar{x}, \bar{y}, z)$ such that for every $m \in \mathbb{N}$, the formula $\Psi^*_P(\bar{x}, \bar{y}, m)$ defines on $\mathbf{P}$ the relation $(R_\Psi)^m$.

If there is such formula, then for every $\bar{s} \in \mathbf{P}$ the set of states in $\mathbf{P}$ reachable from $\bar{s}$ is Presburger definable by the formula $\exists z \Psi^*_P(\bar{s}, \bar{y}, z)$.
Presburger transition systems are rational. We consider the particular case of Presburger definable rational Kripke models, where the valuation is Presburger definable.

**Flat PTS:** every control state belongs to at most one cycle.

[Demri, Finkel, G., van Drimmelen, 2005]:

**Theorem:** In every flat Presburger transition system where every cycle has Presburger definable reachability, the global model checking of modal logic with reachability is decidable.

**Theorem:** In every flat Presburger transition system with functional transition relations, where every cycle has Presburger definable reachability, the global model checking of first-order **CTL** over Presburger arithmetic is decidable.
Regular ground tree rewriting graphs

Ground tree rewriting: extension of word-rewriting to regular languages of finite labelled trees.
Ground tree rewriting systems generate configuration graphs.

Example: the graph generated from $f(c, d)$ by applying the rules $c \rightarrow g(c)$ and $d \rightarrow g(d)$ produces the grid $\mathbb{N} \times \mathbb{N}$.

[Dauchet, Tison, 1990], [Löding, 2002]: Model checking on ground tree rewriting graphs is decidable for FO+ reachability, and for the following fragment of CTL*:

$$\varphi = p \text{ regular} \mid \neg \varphi \mid \varphi_1 \lor \varphi_2 \mid \exists X_a \varphi \mid \exists F \varphi \mid \exists FG \varphi.$$

[Löding, 2002]: Model checking of universal reachability on ground tree rewriting graphs is undecidable.
Connectedness and reachability as modal frame validities

A Kripke frame (directed graph) $F$ is connected iff

$$F \models q \land [U](q \rightarrow [U](q \rightarrow \Box q)) \rightarrow [U]q.$$  

Consider ‘system frames’ $S = (W, R, S_i, S_f)$ where $F = (W, R)$ is a Kripke frame with domain $W$ a regular set in some finite alphabet, and $S_i, S_f$ are two regular subsets of $W$ representing initial and final states.

Add two propositional constants, $s_i$ and $s_f$, to the basic modal language extended with universal modality $[U]$.

Then, in any program frame $S = (W, R, S_i, S_f)$, the set $S_f$ is reachable from $S_i$ iff

$$S \models [U](s_i \rightarrow p) \land [U](p \rightarrow \Box p) \rightarrow \langle U \rangle (p \land s_f).$$

Thus, modal logic over frames is a good restriction of MSO, and classes of infinite frames with decidable modal logic are worth investigating.
Concluding remarks

• The study of finitely presentable infinite structures is still in its status nascendi (excluding some well-studied cases). The underlying idea is to use the finitary presentation to do effective computation on such structures.

• This study involves a highly non-trivial interaction of logic and computation theory, in particular logical interpretations and automata based transformations and manipulations.

• The objective is to analyze the hierarchies of classes of finitely presentable infinite structures and respective logical languages which can be effectively computed on them.

• Combines deep theory and potential for applications.

• Many challenges and open problems, both theoretical and practical.

Ideal field for an ambitious beginning researcher!