Polar Coodinates and Higher Dim Space

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HW2 due Gradescope 9/9

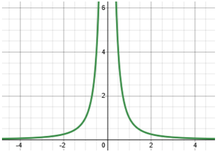
Quiz 9/4 on 10.3, 4, 12.1, 2

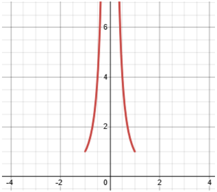
Exercises with section numbers comes from ***Multivariable Calculus*****, Eighth Edition,** James Stewart**.**

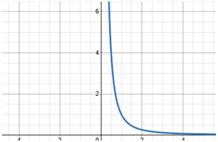
Homework 1

▶ **10.1 Problem 38:** What is the difference between $x = t$, $y = t^{-2}$ vs others?

The difference is range. For example, $\sin t$ or $\cos t$ cannot go past $[-1, 1]$, and e^t cannot be negative.
`https://www.desmos.com/calculator/vk31z1zdzn`







▶ **10.2 Problem 44:** Curve length of $x = 3\cos t - \cos 3t, y = 3\sin t - \sin 3t$.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-3\sin t + 3\sin 3t)^2 + (3\cos t - 3\cos 3t)^2$$

$$= 9\left(\cancel{\sin^2 t} - 2\sin t \sin 3t + \cancel{\sin^2 3t}\cancel{\cos^2 t} - 2\cos t \cos 3t + \cancel{\cos^2 3t} + 2\right)$$

Recall $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ and $\cos(2A) = 2\cos^2(A) - 1$.

$$9(-2\sin t \sin 3t - 2\cos t \cos 3t + 2) = 9(-2\cos(3t - t) + 2) = 18(-\cos(2t) + 1)$$

$$= 18\left(-2\cos^2 t + 1 + 1\right) = 36\left(1 - \cos^2 t\right) = 36\sin^2 t$$

$$L = \int_0^\pi \sqrt{36\sin^2 t}\,dt = \int_0^\pi |6\sin t|dt = \boxed{\int_0^\pi 6\sin t\,dt}$$

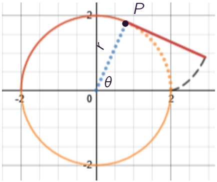
HW 1 - 10.2.73

- **10.2 Problem 53**

Length of $x = a\sin t, y = b\cos t$. That's the circumference of an ellipse - only known solutions are infinite series. Can be a research problem.

- **10.2 Problem 73**

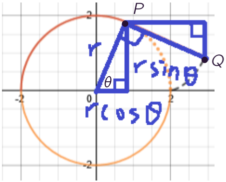
`https://www.desmos.com/calculator/xgggcajmmr`



1. The straight portion of the rope is a tangent line of the circle, and its length equal to the dotted portion of the circle.

2. Radius = r, angle = θ:
   dotted portion has length $r\theta$.

3. Location of point P is $(r\cos\theta, r\sin\theta)$.



4. The two right triangles are similar triangles, both are $1 - sin\theta - \cos\theta$, going counterclockwise,

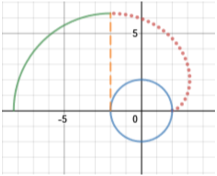
5. So Q is $r\theta\sin\theta$ to the right of P and $r\theta\cos\theta$ below P.

6. Location of point Q is $(r\cos\theta + r\theta\sin\theta, r\sin\theta - r\theta\cos\theta)$.

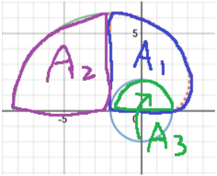
HW 1 - 10.2.74

- 10.2 Problem 74

Assume you got $x = r(\cos\theta + \theta\sin\theta)$, $y = r(\sin\theta - \theta\cos\theta)$.



We need to find this area, times 2, because the cow can move upwards or downwards.



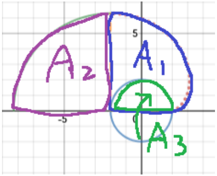
We label the areas A_1, A_2, A_3.
We want to calculate $A_1 + A_2 - A_3$,
and A_2, A_3 are semicircles.
(Poll) Suppose the radius of A_3 is R. What is the radius of A_2?

This is not polar coordinate. The angle θ refers to the last point on the barn where the rope still touches the barn. So the cow starts at $\theta = 0$ and enters A_2 when $\theta = \pi$.

$$A_1 = \left| \int_{\theta=0}^{\theta=\pi} y dx \right|$$

HW 1 - 10.2.74

- 10.2 Problem 74



We label the areas A_1, A_2, A_3.
We want to calculate $A_1 + A_2 - A_3$,
and A_2, A_3 are semicircles.

Plug in $x = r(\cos\theta + \theta\sin\theta)$, $y = r(\sin\theta - \theta\cos\theta)$:

$$\begin{aligned}
A_1 &= \left|\int_{\theta=0}^{\theta=\pi} y dx\right| \\
&= \left|\int_{\theta=0}^{\theta=\pi} y x'(\theta) d\theta\right| \\
&= \left|\int_{\theta=0}^{\theta=\pi} r(\sin\theta - \theta\cos\theta) r(\cancel{-\sin\theta} + \theta\cos\theta + \cancel{\sin\theta}) d\theta\right| \\
&= r^2 \left|\int_{\theta=0}^{\theta=\pi} (\theta\sin\theta\cos\theta - \theta^2\cos^2\theta) d\theta\right|
\end{aligned}$$

Integration by parts:

$$\int \theta\sin\theta\cos\theta d\theta = \frac{1}{2}\int \theta\sin 2\theta d\theta = \frac{1}{4}\int -\theta d\cos 2\theta = -\frac{1}{4}\theta\cos 2\theta - \frac{1}{4}\int -\cos 2\theta d\theta \tag{1}$$

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2} \implies \int \theta^2 \frac{\cos 2\theta + 1}{2} d\theta = \frac{1}{4}\int \theta^2 d\sin 2\theta + \frac{\theta^3}{6}$$

$$= \frac{1}{4}\theta^2\sin 2\theta - \frac{1}{4}\int \frac{\sin 2\theta}{2} 2\theta d\theta + \frac{\theta^3}{6} \quad \text{Then use (1)}$$

Section 10.3, 10.4

Find polar coordinates of points. 10.3: 1,3.

Plot regions in polar coordinates. 10.3: 7,9,11.

Find Cartesian equation of polar curves. 10.3: 15, 16, 17, 18, 19, 20.

Sketch curve from polar equation. 10.3: 29, 33, 35, 43, 45.

Find tangent line at a point. 10.3: 57, 60.

Solve for horizontal tangents. 10.3: 63.

Area inside polar curve. 10.4: 1, 3, 9, 11, 17.

Length of polar curve. 10.4: 45, 47

- The non boldface ones follows from the definition.
- Since equations in polar coordinates are generally of the form
$$r = f(\theta),$$
you can rederive the formula for the boldface ones by converting it into
$$x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$$
and use corresponding results from parametric equations.

$$\boxed{A = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta = \int_a^b \frac{1}{2}r^2\,d\theta} \quad \boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta}$$

▶ **10.3 Problem 58** Find the tangent line of $r = \cos(\theta/3)$ at $\theta = \pi$.

$$\frac{dx}{d\theta} = \frac{d\cos(\theta/3)\cos\theta}{d\theta} = -\frac{1}{3}\sin(\theta/3)\cos\theta - \cos(\theta/3)\sin\theta \qquad \left.\frac{dx}{d\theta}\right|_{\theta=\pi} = \frac{\sqrt{3}}{6}$$

$$\frac{dy}{d\theta} = \frac{d\cos(\theta/3)\sin\theta}{d\theta} = -\frac{1}{3}\sin(\theta/3)\sin\theta + \cos(\theta/3)\cos\theta \qquad \left.\frac{dy}{d\theta}\right|_{\theta=\pi} = -\frac{1}{2}$$

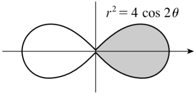
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-1/2}{\sqrt{3}/6} = -\frac{1}{\sqrt{3}/3} = -\sqrt{3}$$

When $\theta = \pi$, we are at $\left(-\frac{1}{2}, 0\right)$, so the line is

$$y = -\sqrt{3}\left(x + \frac{1}{2}\right)$$

Area and Length of Polar Curves

▶ **10.4 Problem 18** Find the area enclosed by one loop of $r^2 = 4\cos 2\theta$.



We know the loop closes when $r = 0$,
and that is when $\cos 2\theta = 0 \iff 2\theta = \frac{\pi}{2} + n\pi$
$\iff \theta = \frac{\pi}{4} + n\frac{\pi}{2}$

So our bounds of integration would be $\frac{\pi}{4} + n\frac{\pi}{2}$ with two consecutive integer, say $n = -1$ and $n = 0$.

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2}\left(4\cos 2\theta\right) d\theta = \Big[\sin 2\theta\Big]_{-\pi/4}^{\pi/4} = 2$$

▶ **10.4 Problem 46** Find the length of the curve $r = 5^\theta, 0 \leq \theta \leq \pi$.

$$\begin{aligned} L &= \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\ &= \int_0^\pi \sqrt{5^{2\theta} + \left((\ln 5)5^\theta\right)^2} d\theta \\ &= \int_0^\pi 5^\theta \sqrt{1 + (\ln 5)^2} d\theta \\ &= \sqrt{1 + (\ln 5)^2} \int_0^\pi 5^\theta d\theta \end{aligned}$$

3D space and Vectors, 12.1, 12.2

Plotting point/line/spheres in Cartesian coordinates 12.1: 1, 3, 5, 7, 10, 11, 12

Equation of spheres in Cartesian coordinates 13, 15, 16, ~~19a-b~~ [maybe he meant 21a-b], 22.

Adding vectors, drawing vectors 12.2: 1a-d, 5, 13, 14, 15, 18, 19, 32.

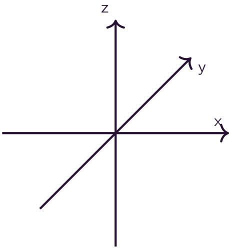
Finding unit vectors of the same direction. 12.2: 24, 25, 26, 27, 41.

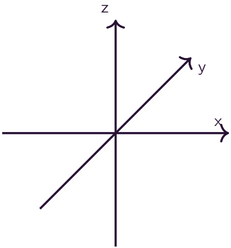
Word problem with vectors 12.2: 35.

What point is $(4, 5, -3)$?

What point is $2 \cdot (0, 0, 1) + 6 \cdot (0, 1, 0) + (-3) \cdot (1, 0, 0)$?

What unit vector is parallel to $2 \cdot (0, 0, 1) + 6 \cdot (0, 1, 0) + (-3) \cdot (1, 0, 0)$?





Sphere in 3 coordinates

▶ **12.1 Problem 14** Find an equation of the sphere with center $(2, -6, 4)$ and radius 5. Describe its intersection with each of the coordinate planes.

$$(x-2)^2 + (y+6)^2 + (z-4)^2 = 25$$

$$\begin{aligned} x = 0 &\implies 4 + (y+6)^2 + (z-4)^2 = 25 \\ y = 0 &\implies (x-2)^2 + 36 + (z-4)^2 = 25 \\ z = 0 &\implies (x-2)^2 + (y+6)^2 + 16 = 25 \end{aligned}$$

▶ **12.1 Problem 23** Find equations of the spheres with center (2, -3, 6) that touch (a) the xy-plane, (b) the yz-plane, (c) the xz-plane.

What is the distance of the center to each of the three planes?

Observation: when radius of a sphere equals its distance to an object, then the sphere touches the object.

$$(x-2)^2 + (y+3)^2 + (z-6)^2 = \text{distance to plane}^2$$

n Dimensional space, looking like a line locally.

- **What is a plane in higher dimensions?** There are at least two natural generalizations, inspired by our $\mathbb{R}^3$ intuitions.
  - An object with 2 dimensions, i.e. all points in it can be expressed as $a\vec{u} + b\vec{w}$, $a, b \in \mathbb{R}$.
  - An object with 1 dimension less than the entire space, i.e. can be expressed as

    $$3x + 4y + 5z = -1 \quad \text{or} \quad a_1x_1 + a_2x_2 + \ldots + a_nx_n = b, a_i, b \in \mathbb{R}.$$

  - In general, the intersection of two 2-dimensional space in $\mathbb{R}^3$ is a line (the parallel case is a special case)
  - In general, the intersection of two 2-dimensional space in $\mathbb{R}^4$ is a point.
- **Looking like a line locally.** Notice this curve below has a unique tangent line everywhere except at $(0, 0)$. You can rephrase that as this curve looks like a line, or $\mathbb{R}^1$, locally, except at $(0, 0)$.

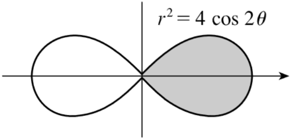


Figure: From 10.4.18