0916 - DIS 204

Tuesday, September 15, 2020 7:39 PM

1. Find all partials and second partials of $f(x, y) = 3xy - x^2 - y^2$ at the origin.

$f(x,y) = f(y,x)$

$f_x = 3y - 2x \quad f_y = 3x - 2y$

Polynomial in x and y $\Rightarrow f_x \ f_y$ continuous on all $\mathbb{R}^2 \ni (x,y)$

f is differentiable on all of $(x,y) \in \mathbb{R}^2$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} f_x = -2 = \frac{\partial^2 f}{\partial y^2} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} f_y = 3 = \frac{\partial^2 f}{\partial y \partial x}$$

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2. What is the plane tangent to $f(x, y) = x^3 - 3xy^2$ at $(2, 1, 2)$?

$$\frac{\partial f}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial f}{\partial y} = -6xy$$

$\longrightarrow$ both continuous on all $\mathbb{R}^2$

at $(2,1,2)$ $f_x = 9$ $f_y = -12$

$$\boxed{9(x-2) + (-12)(y-1) - (z-2) = 0}$$

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3. What is the plane tangent to

$$f(x, y) = \frac{7xy}{e^{x^2+y^2}}$$

(1) at $x = 1, y = 1$?

(2) When is the tangent plane parallel to the xy-plane?

$$f(x,y) = \frac{7xy}{e^{x^2+y^2}} = \frac{7yx}{e^{y^2+x^2}} = f(y,x)$$

(1)
$$\frac{\partial f}{\partial x} = \frac{e^{x^2+y^2} 7y - 7xy \cdot 2x e^{x^2+y^2}}{e^{2(x^2+y^2)}}$$

$$\frac{\partial f}{\partial y} = \frac{e^{x^2+y^2} 7x - 7yx \cdot 2y e^{x^2+y^2}}{e^{2(x^2+y^2)}}$$

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial f}{\partial \text{ 1st coord.}} = \frac{\partial f(y,x)}{\partial \text{ 1st coord}}\Big|_{\text{exchange } x,y} = \frac{\partial f}{\partial y}\Big|_{\text{exchange } x,y}$$

$x = y = 1$ $\quad \frac{\partial f}{\partial x} = \frac{-7}{e^2} = \frac{\partial f}{\partial y}$

$$\boxed{\frac{-7}{e^2}(x-1) + \frac{-7}{e^2}(y-1) - \left(z - \frac{7}{e^2}\right) = 0}$$

(2) $e^{2(x^2+y^2)} \neq 0$ $\quad f_x = e^{x^2+y^2} \cdot 7y\,(1-2x^2) = 0$

$$f_y = e^{x^2+y^2} \cdot 7x\,(1-2y^2) = 0$$

$$\begin{cases} x = 0 \quad y = 0 \\ (1-2x^2) = 0 = (1-2y^2) \\ \hookrightarrow \; x = \pm \frac{\sqrt{2}}{2} \\ \quad\;\; y = \pm \frac{\sqrt{2}}{2} \end{cases}$$

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4. If $z(x, y) = x^3 - 3xy^2$ and $x = t^2 - w^2$, $y = t + w$, what is $\frac{\partial z}{\partial t}$? What is $\frac{\partial z}{\partial w}$?

$$\begin{tikzcd} t \arrow[r] \arrow[dr] & x \arrow[dr] & \\ & & z \\ w \arrow[uur] \arrow[r] & y \arrow[ur] & \end{tikzcd}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = (3x^2 - 3y^2)\cdot 2t + (-6xy)\cdot 1$$

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial w}$$

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5. If $f(x, y) = xy^2 - x^2y$, $y(x, t) = t + \sin x$, what is $\frac{\partial f}{\partial x}$?

What is $\frac{\partial^2 f}{\partial x^2}$?

$$\text{"}\frac{\partial f}{\partial x}\text{"} = \lim_{h \to 0} \frac{f(x+h, y(x+h, t)) - f(x, y(x,t))}{x+h-x}$$

$$\begin{tikzcd} x \arrow[d] \arrow[dr] & \\ y \arrow[r] & f \\ t \arrow[u] & \end{tikzcd}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}\Big|_{y \text{ fixed}} + \frac{\partial f}{\partial y}\Big|_{x \text{ fixed}} \cdot \frac{\partial y}{\partial x}\Big|_{t \text{ fixed}}$$

$$= \boxed{y^2 - 2xy + (2xy - x^2)\cdot\cos x}$$

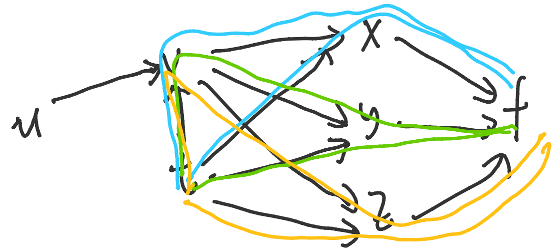
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)_{y \text{ fixed}} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_{x \text{ fixed}} \left(\frac{\partial y}{\partial x}\right)\Big|_{t \text{ fixed}}$$

$$f(x, t + \sin x) = x(t + \sin x)^2 - x^2(t + \sin x)$$

$$\frac{\partial f}{\partial x} = \underbrace{x \cdot 2(t + \sin x)\cos x}_{2xy\cos x} + \underbrace{(t + \sin x)^2}_{y^2}$$

$$\underbrace{-x^2 \cdot \cos x}_{-x^2\cos x} - \underbrace{2x(t + \sin x)}_{-2xy}$$

6. If we have $f(x,y,z)$, $x(v,t)$, $y(v,t)$, $z(v,t)$, $v(u,t)$, what does the dependencies look like? What is $\frac{\partial f}{\partial t}$?



$$\frac{\partial f}{\partial t} = \left.\frac{\partial f}{\partial x}\right|_{y,z} \cdot \left.\frac{\partial x}{\partial v}\right|_{t} \left.\frac{\partial v}{\partial t}\right|_{u} + \left.\frac{\partial f}{\partial x}\right|_{y,z} \cdot \left.\frac{\partial x}{\partial t}\right|_{v} + \text{replace } x \text{ by } y$$

$$+ \text{ replace } x \text{ by } z$$

"Application of partial derivatives"

$F(t) = \int f(x,t)\,dx$ ← after integration, x is gone

Suppose $f(x,t)$ integrable (t constant)

$\frac{\partial}{\partial t} f(x,t)$ $\begin{cases} \text{exists } (x \text{ constant}) \\ \text{integrable wrt } x \end{cases}$

$$\frac{\partial}{\partial t} F(t) = \int \frac{\partial}{\partial t} f(x,t)\,dt$$

$u = (t-x)$

☆ $F(t) = \int f(t-x)\, g(x)\, dx = \int f(u)\, g(t-u)\, du$

$$\frac{\partial}{\partial t} F(t) = \int \frac{\partial f(t-x)}{\partial t} g(x)\, dx = \int f(u) \cdot \frac{\partial}{\partial t} g(t-u)\, du$$

↑

$F(t)$ differentiable if g differentiable