0921 - DIS 202

Sunday, September 20, 2020 11:55 PM

Assume all functions are differentiable.

Here $f(x, y)$ means it maps from $\mathbb{R}^2 \to \mathbb{R}$, and $f(x, y, z)$ means it maps from $\mathbb{R}^3 \to \mathbb{R}$.

1. (a) $D_u f(x, y)$ is maximized when $\vec{u}$ points opposite to $\vec{\nabla} f(x, y)$.

   (b) For $f(x, y)$, there is always a direction $\vec{u}$ where $D_u f(x, y)$ is zero.

(a) $D_u f = \vec{\nabla} f \cdot \vec{u}$ abs value maximized $\vec{\nabla} f \parallel \vec{u}$

$|D_u f| = |\vec{\nabla} f| \cdot |\vec{u}| \cdot |\cos\theta| \to \cos\theta = \pm 1$

$\vec{u}$ opposite : $D_u f = -|\vec{\nabla} f| \cdot |\vec{u}|$ "large but negative"

$\boxed{\text{false}}$

(b) $\theta = \pm\frac{\pi}{2}$ orthogonal $\vec{u} \perp \vec{\nabla} f \Rightarrow \vec{\nabla} f \cdot \vec{u} = 0$

$\boxed{\text{True}}$

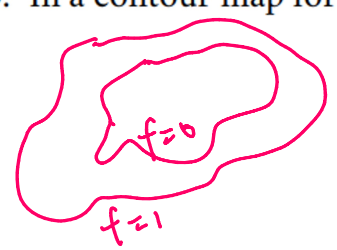
2. For $f(x, y, z)$, if $\vec{\nabla} f(x, y, z) \cdot \vec{v} = 0$ for all $\vec{v} \in \mathbb{R}^3$, then f_x, f_y, f_z are all zero there.

$f_x = \underbrace{\vec{\nabla} f(x, y, z)}_{\langle f_x, f_y, f_z \rangle \cdot \mathbf{i}} \cdot \vec{v}$, when $\vec{v} = \langle 0, 0, 0 \rangle$: no direction

$\vec{v} = \mathbf{i} = \langle 1, 0, 0 \rangle$ $\boxed{\text{True}}$

$f_y = \vec{\nabla} f(x, y, z) \cdot \langle 0, 1, 0 \rangle$ $f_z = \vec{\nabla} f(x, y, z) \cdot \langle 0, 0, 1 \rangle$

3. In a contour map for $f(x, y)$, $\vec{\nabla} f(x, y)$ is parallel to the level curves.



$\boxed{\text{False}}$

1(a) $\vec{\nabla} f(x,y)$ & $\vec{u}$ points same dir

$D_u f$ maximized

$\vec{\nabla} f(x,y)$ = direction f change fastest

1(b) $\vec{\nabla} f$ perpendicular to $D_u f = 0$ direction

$\Leftrightarrow$ perpendicular to level curves

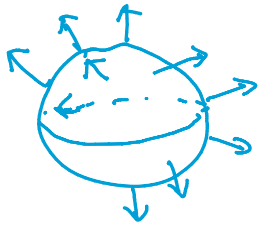
Extra:

$f(x, y, z)$ define surface $f(x, y, z) = k$

$\vec{\nabla} f(x, y, z)$ on surface : $\vec{\nabla} f$ $\left\{ \begin{array}{l} \text{perpendicular} \\ \text{normal} \end{array} \right.$ to surface

$f(x, y, z) = x^2 + y^2 + z^2$

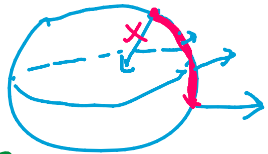
$\to f(x, y, z) = k$ $\vec{\nabla} f = \langle f_x, f_y, f_z \rangle = \langle 2x, 2y, 2z \rangle$



"outward" $\boxed{\text{length of } \vec{\nabla} f \text{ constant}}$

$\vec{\nabla} f$ changes "continuously" in terms of x, y, z

i.e. x, y, z change a little $\Longleftrightarrow$ $\vec{\nabla} f$ change a little



cannot just flip

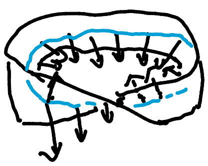
$\vec{\nabla} f$ has to be zero somewhere

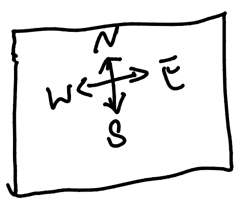
if it points outside $\xrightarrow{\text{then}}$ inside

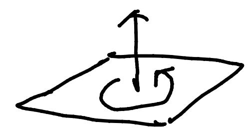
$g(x,y,z) = -x^2 - y^2 - z^2$

$g(x,y,z) = -k$ all surface normal given by $\vec{\nabla} f$ point inward

"Orientability": Able to choose a continuous surface normal consistently







"choose clockwise/counterclockwise directions consistently"

if you comb a sphere there must be cowlick

40. (a) If $\mathbf{u} = \langle a, b \rangle$ is a unit vector and f has continuous second partial derivatives, show that

$$D_{\mathbf{u}}^2 f = f_{xx} a^2 + 2 f_{xy} ab + f_{yy} b^2$$

$f_{xy} = f_{yx}$

(b) Find the second directional derivative of $f(x, y) = xe^{2y}$ in the direction of $\mathbf{v} = \langle 4, 6 \rangle$.

(a) $D_{\vec{u}}(D_{\vec{u}} f) = \vec{\nabla} D_{\vec{u}} f \cdot \vec{u} = \langle (af_x + bf_y)_x, (af_x + bf_y)_y \rangle \cdot \vec{u}$

$D_u f = \vec{\nabla} f \cdot \vec{u} = \langle f_x, f_y \rangle \cdot \langle a, b \rangle = af_x + bf_y$

$D_{\vec{u}}(D_{\vec{u}} f) = \langle af_{xx} + bf_{yx}, af_{xy} + bf_{yy} \rangle \cdot \langle a, b \rangle$

$= a^2 f_{xx} + ab f_{yx} + ab f_{xy} + b^2 f_{yy}$