0921 - DIS 204

Sunday, September 20, 2020 11:55 PM

Assume all functions are differentiable.

Here $f(x, y)$ means it maps from $\mathbb{R}^2 \to \mathbb{R}$, and $f(x, y, z)$ means it maps from $\mathbb{R}^3 \to \mathbb{R}$.

1. (a) $D_u f(x, y)$ is maximized when $\vec{u}$ points opposite to $\vec{\nabla} f(x, y)$.

   (b) For $f(x, y)$, there is always a direction $\vec{u}$ where $D_u f(x, y)$ is zero.

1. (a) $D_u f(x,y) = \vec{\nabla} f \cdot \vec{u}$ ($\vec{u}$ is a unit vec)

$= |\vec{\nabla} f| \cdot |\vec{u}| \cdot \cos\theta$ $\boxed{\text{False}}$

$\cos\theta = +1$ $D_u f(x,y)$ maximized large & positive

$\cos\theta = -1$ $|D_u f(x,y)|$ also maximized, $D_u f(x,y)$ large & negative

(b) $|D_u f(x,y)| = |\vec{\nabla} f| \cdot |\vec{u}| \cdot |\cos\theta|$

$\cos\theta = 0$, $\theta = \pm\frac{\pi}{2}$ $\Rightarrow$ $D_u f(x,y) = 0$ Direction where $D_u f = 0$ is perpendicular to $\vec{\nabla} f$

$\boxed{\text{True}}$

2. For $f(x, y, z)$, if $\vec{\nabla} f(x, y, z) \cdot \vec{v} = 0$ for all $\vec{v} \in \mathbb{R}^3$, then f_x, f_y, f_z are all zero there.

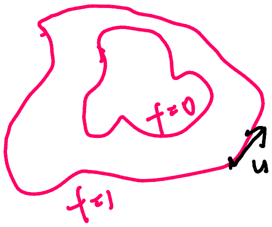
$i = \vec{v}_1 = \langle 1,0,0 \rangle$ $\vec{v}_2 = \langle 0,1,0 \rangle$ $\vec{v}_3 = \langle 0,0,1 \rangle$

$f_x = \langle f_x, f_y, f_z \rangle \cdot \langle 1,0,0 \rangle = \vec{\nabla} f \cdot \vec{v}_1 = D_{\vec{v}_1} f = 0$

$f_y = \langle f_x, f_y, f_z \rangle \cdot \langle 0,1,0 \rangle = \vec{\nabla} f \cdot \vec{v}_2 = 0$ $\boxed{\text{True}}$

$f_z = \vec{\nabla} f \cdot \langle 0,0,1 \rangle = 0$

3. In a contour map for $f(x, y)$, $\vec{\nabla} f(x, y)$ is parallel to the level curves.



$D_u f = 0$ if $u \parallel$ curve

1(a) $\vec{\nabla} f(x,y)$ parallel to direction f changes the fastest

1(b) Along level curves, $D_u f = 0$

$\Leftrightarrow$ level curves are perpendicular to $\vec{\nabla} f$

Generalize:

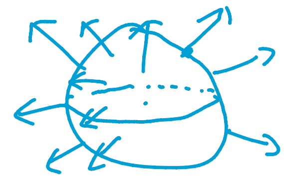
$f(x,y,z)$ level surfaces $f(x,y,z) = k$

1(b) if $\vec{\nabla} f \cdot \vec{u} = 0$ then $\vec{u}$ perpendicular to $\vec{\nabla} f$

$\vec{\nabla} f$ gives a vector $\left\{ \begin{array}{l} \text{perpendicular} \\ \text{normal} \end{array} \right.$ to the level surface

$f(x,y,z) = x^2+y^2+z^2$

$\vec{\nabla} f = \langle 2x, 2y, 2z \rangle$

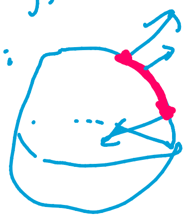


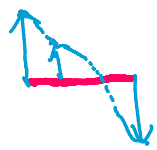
$x^2+y^2+z^2 = k \quad k \in (0, \infty)$

$\vec{\nabla} f = \langle 2x, 2y, 2z \rangle$ change "continuously" in terms of x,y,z

i.e. $\langle 2x, 2y, 2z \rangle$ changes a little $\Leftarrow$ x,y,z change a little

If $\vec{\nabla} f$ flipped:





continuous: cross zero

$|\vec{\nabla} f| = \sqrt{(2x)^2+(2y)^2+(2z)^2}$

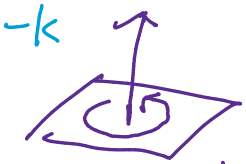
$= \sqrt{4(x^2+y^2+z^2)}$

$= \sqrt{4k}$

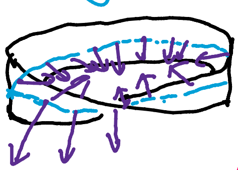
"Able to make a consistent choice of a perpendicular vector" $\rightarrow$ Orientability

$g(x,y,z) = -x^2-y^2-z^2 \qquad g(x,y,z) = -k$

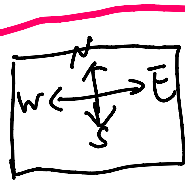
$\vec{\nabla} g = \langle -2x, -2y, -2z \rangle$



able to make a consistent choice of clockwise/counter clockwise



no consistent choice



$\rightarrow$ arrows are along surface

40. (a) If $\mathbf{u} = \langle a, b \rangle$ is a unit vector and f has continuous second partial derivatives, show that

$$D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$$

$f_{xy} = f_{yx}$

(b) Find the second directional derivative of $f(x, y) = xe^{2y}$ in the direction of $\mathbf{v} = \langle 4, 6 \rangle$.

14.6.39 $D_u(D_u f)$

$D_u f(x,y) = \vec{\nabla} f \cdot \vec{u} = \langle f_x, f_y \rangle \cdot \langle a, b \rangle$

$= af_x + bf_y$

$D_u(D_u f) = \vec{\nabla} D_u f \cdot \vec{u} = \langle af_{xx} + bf_{yx}, af_{xy} + bf_{yy} \rangle \cdot \langle a, b \rangle$

$= a^2 f_{xx} + abf_{yx} + baf_{xy} + b^2 f_{yy}$