# 0923 - DIS 202

Tuesday, September 22, 2020  11:49 PM

We assume the 2nd partial derivatives of $f(x, y)$ are continuous on a disk around $(a, b)$.

When $f_x(a, b) = f_y(a, b) = 0$, we define

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - \left(f_{xy}(a, b)\right)^2$$

(a) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$ , then $(a, b)$ is a local minimum.

(b) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$ , then $(a, b)$ is a local maximum.

(c) If $D(a, b) < 0$ then it is neither a local minimum nor a local maximum.

## True/False.

1. If $\underline{D(a, b) > 0}$ and $f_{xx}(a, b) > 0$, then $f_{yy}(a, b) > 0$. $\boxed{\text{True}}$

$f_{xx} f_{yy} - f_{xy}^2 > 0$   $\underline{f_{xx} f_{yy}} > f_{xy}^2 \geq \underline{0}$   $f_{xx} f_{yy} > 0$ & $f_{xx} > 0$

$\Rightarrow f_{yy} > 0$

2. If $\underline{D_u(D_u f(a, b)) > 0 \text{ in every direction } u}$, then $(a, b)$ is a local minimum.

implies $\underline{D(a,b) > 0}$ & $f_{xx} > 0$ and $f_x = 0$ $f_y = 0$

hard   $D_u{}^2 f > 0$   $\Rightarrow D_u f = 0$   $\boxed{\text{True}}$

along $u$   $D_u{}^2 f > 0$

Similarly, $\left\{\begin{array}{l} D_u{}^2 f > 0 \\ D_v{}^2 f < 0 \end{array}\right.$ at $(a,b)$ while $f_x = f_y = 0$ shows saddle point

3. If $D(a, b) = 0$, then $f$ can have a maximum, minimum, or a saddle point at $(a, b)$.

Example: $-\underline{D(a,b) = 0, \text{ maximum}}$:

SV Calculus: $f(x) = x^4$  $f''(x) = 0$ 

$f(x,y) = x^4 + y^4$   $f_x = 4x^3 = 0$ at $(0,0)$

$f_y = 4y^3 = 0$

$f_{xy} = 0$   $f_{xx} = 12x^2 = 0$

$f_{yy} = 12y^2 = 0$

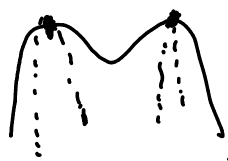
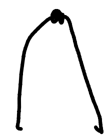
$-\underline{D(a,b) = 0 \text{ min}}$: $g(x,y) = -f(x,y)$

$-\ D(a,b)=0$ neither max nor min:

$h(x,y)=c$

4. Give a function with exactly two local maxima, and find all of its critical points.

$f(x,y)=-(x-1)^2(x+1)^2-y^2=-(x^2-1)^2-y^2$

along x:  along y: 

$f_x=-2(x^2-1)\cdot 2x=-4x^3+4x$ $\qquad f_y=-2y$

$f_x=0:\ x=\pm 1, 0$ $\qquad f_y=0:\ y=0$

$(1,0)\quad(-1,0)\qquad(0,0)$

$f_{xx}=-12x^2+4 \qquad f_{xy}=0 \qquad f_{yy}=-2$

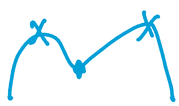
$D(1,0)=(-8)\cdot(-2)-0^2=16$

$D(-1,0)=16$

$D(0,0)=4\cdot(-2)-0^2=-8<0$

Max: $(1,0)\ (-1,0)$ $\qquad$ Neither: $(0,0)$

$f(x,y)=-(x^2-1)^2+y^2$

along x  along y 

Neither $(1,0)\ (-1,0)$ $\qquad$ min $(0,0)$

$g(x,y)=-(x^2-1)^2-(x^2y-x-1)^2$

$g_x=-4x^3+4x-2(x^2y-x-1)(2xy-1)=0$

$g_y=-2(x^2y-x-1)x^2=0$

$\to\ x=0\ \to\ g_x=-2\cdot(-1)(-1)=-2\neq 0$

$(x^2y-x-1)=0\ \to\ \boxed{\text{want } g_x=0}$ $\qquad x^3-x=0 \qquad x=\pm 1, 0$

$\cancel{x=0}$ $\qquad x^2y-x-1=-1\neq 0$

$$\begin{cases} x = 1 & x^2y - x - 1 = y - 2 = 0 \\ x = -1 & x^2y - x - 1 = y = 0 \end{cases}$$

$g(x,y)$ has two critical pts $(1,2)$ $(-1,0)$

Also, $g(x,y) \le 0$, only attains $0$ at

$x^2 - 1 = 0$ and $x^2y - x - 1 = 0$

which is $(1,2)$ $(-1,0$

5. Give a function with exactly one saddle point, and show this is the case.

$$f(x,y) = x^2 - y^2$$

$$f_x = 2x \quad f_{xx} = 2$$

$$f_y = -2y \quad f_{yy} = -2 \qquad D(0,0) = 2\cdot(-2) - 0^2 = -4 < 0$$

$$f_{xy} = 0$$

$$f_x = f_y = 0 \implies (0,0)$$

6. Give a function with more than one saddle point and no local minima or maxima.

$$f(x,y) = 4xy^2 - 2x^2y - x$$

$$f_x = 4y^2 - 4xy - 1$$

$$f_y = 8xy - 2x^2 = 2x(4y - x)$$

$$f_x = f_y = 0$$

$$x = 0,\ 4y^2 = 1 \quad \text{or} \quad 4y - x = 0,\ \cancel{-12y^2 - 1 = 0}$$

$(0, \frac{1}{2})$ $(0, -\frac{1}{2})$ critical pts

$$f_{xx} = -4y \quad f_{xy} = 8y - 4x \quad f_{yy} = 8x$$

$$D(0,\tfrac{1}{2}) = -2\cdot 0 - 4^2 = -16 < 0$$

$$D(0,-\tfrac{1}{2}) = 2\cdot 0 - 4^2 = -16 < 0$$

two saddles