0923 - DIS 204

Tuesday, September 22, 2020 11:49 PM

We assume the 2nd partial derivatives of $f(x, y)$ are continuous on a disk around (a, b).

When $f_x(a,b) = f_y(a,b) = 0$, we define

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

(a) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum.

(b) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local maximum.

(c) If $D(a,b) < 0$ then it is neither a local minimum nor a local maximum.

True/False.

1. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f_{yy}(a,b) > 0$.

$\boxed{\text{True}}$

$f_{xx} f_{yy} - f_{xy}^2 > 0$

$f_{xx} f_{xy} > f_{xy}^2 \geq 0 \longrightarrow f_{xx} f_{yy} > 0$ & $f_{xx} > 0$

$\rightarrow f_{yy} > 0$

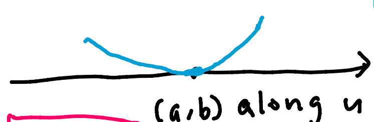
2. If $D_u(D_u f(a,b)) > 0$ in every direction u, then (a,b) is a local minimum.

$D_u^2 f > 0$

prototype $f''(x) > 0$: x^2

$f_x = f_y = 0$

$\rightarrow D_u f = 0$



curve upwards along every $u \longrightarrow$ minimum

$\boxed{\text{True}}$

Saddle pts: neither max nor min, has values strictly $> f(a,b)$ & strictly $< f(a,b)$ in any region around (a,b)

could find $\vec{u}, \vec{v}$ s.t. $D_u^2 f > 0$, $D_v^2 f < 0$ & $f_x = f_y = 0$

then it's a saddle pt.

$D(a,b) = D_u^2 f \cdot D_v^2 f$ (2)

where $D_u^2 f = \max\limits_{|w|=1} D_w^2 f \quad D_v^2 f = \min\limits_{|w|=1} D_w^2 f$

$w = \langle \cos\theta, \sin\theta \rangle \quad D_w^2 f = F(\theta), \ \theta \in [0, 2\pi]$

continuous function closed interval

$F(\theta)$ attains its max & min in the interval

$D(a,b) > 0$, given interpretation (2),

$D_w^2 f$ has the same sign for all $|w| = 1$

$f_{xx} > 0 \rightarrow D_w^2 f > 0 \ \forall w$

$f_{xx} < 0 \rightarrow D_w^2 f < 0 \ \forall w$

$D(a,b) < 0$, given (2),

along u, $D_u^2 f > 0$ $\cup$ → saddle

along v, $D_v^2 f < 0$ $\cap$

3. If $D(a, b) = 0$, then f can have a maximum, minimum, or a saddle point at (a, b).

Example: $\boxed{\text{True}}$

- $D(a,b) = 0$, maximum:

SV Calculus $f(x) = -x^4$ $f'(x) = 0$ at $x = 0$

$f''(x) = -12x^2 = 0$

$f(x,y) = -x^4 - y^4$ max at $(0,0)$

$f_x = -4x^3$ $f_y = -4y^3$ $f_{xx} = -12x^2$ $f_{yy} = -12y^2$ $f_{xy} = 0$

$D(0,0) = 0 \cdot 0 - 0^2 = 0$

- $D(a,b) = 0$, minimum:

$g(x,y) = -f(x,y)$

- $D(a,b) = 0$, saddle:

ordinary saddle $p(x,y) = xy$

$h(x,y) = x^3 y^3$

$h_x = 3x^2y^3 \quad h_y = 3x^3y^2 \quad h_{xx} = 6xy^3 \quad h_{yy} = 6x^3y \quad h_{xy} = 9x^2y^2$

$(x,y) = (0,0) \qquad = 0 \qquad = 0 \qquad = 0$

$D(0,0) = 0^2 - 0^2 = 0$

4. Give a function with exactly two local maxima, and find all of its critical points.

((CalcPlot3D)

$$e^{-((x-1)^2+y^2)} + e^{-((x+1)^2+y^2)}$$

$-(x-1)^2(x+1)^2 - y^2$ 2 max 1 saddle

$-(x-1)^2(x+1)^2 + y^2$ 2 saddle 1 min

$f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2$

$f_x = -4x^3 + 4x - 2(x^2y - x - 1)(2x - 1)$

$f_y = -2(x^2y - x - 1)\,x^2 \rightarrow x = 0$ or $(x^2y - x - 1)$

$f_x = f_y = 0$ $\qquad f_x \neq 0 \qquad f_x = 0 \rightarrow x = \pm 1, \cancel{x}$

$(x,y) = (1,2) \quad (-1,0)$ critical pts

$f(x,y) \leq 0$ $\quad f(x,y)$ attains a max of 0 when

$x^2 - 1 = 0$ & $x^2y - x - 1 = 0$

maxima: $(1,2) \quad (-1,0)$

5. Give a function with exactly one saddle point, and show this is the case.

$f(x,y) = xy$ or $g(x,y) = x^2 + y^2 \leftarrow$ essentially rotation of f

$\underline{f_x = y \quad f_y = x}$ critical pt $y = x = 0$

$f_{xx} = f_{yy} = 0$

$f_{xy} = 1$

$D(0,0) = 0 \cdot 0 - 1^2 = -1$

6. Give a function with more than one saddle point and no local minima or maxima.

$f(x,y) = 4xy^2 - 2x^2y - x$

$f_x = 4y^2 - 4xy - 1 \qquad f_x = f_y = 0$

$f_y = 8xy - 2x^2$

$\rightarrow x = 0 \quad \text{or} \quad y = \frac{1}{4}x$

$4y^2 = 1 \qquad \cancel{-12y^2 - 1} = 0$

$(0, \frac{1}{2}) \qquad (0, -\frac{1}{2}) \quad$ critical pts

$f_{xx} = -4y \quad f_{xy} = 8y - 4x \quad f_{yy} = 8x$

$D(0, \frac{1}{2}) = -2 \cdot 0 - 4^2 = -16 < 0$

$D(0, -\frac{1}{2}) = 2 \cdot 0 - 4^2 = -16 < 0$

two saddles