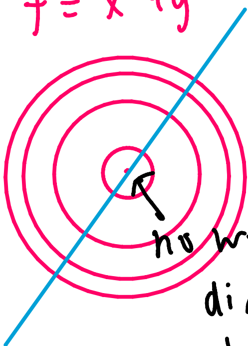
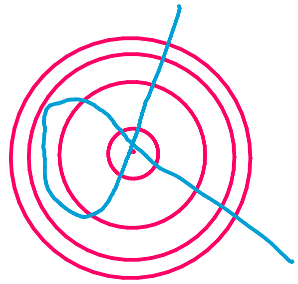
1002 - DIS 204

Wednesday, September 30, 2020 8:01 AM

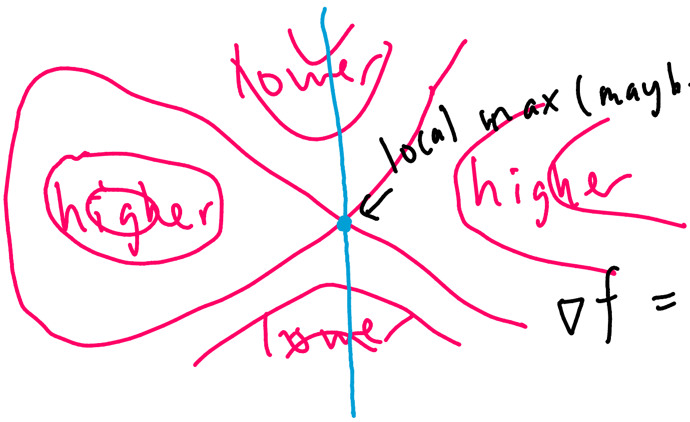
1. If $f(x, y)$ is maximized at (x_0, y_0) given the constraint $g(x, y) = k$, then the level curve of f at (x_0, y_0) is parallel to $g(x, y) = k$ there.

$f = x^2 + y^2$ $g = y = x$





no well defined direction for level curves of f



local max (maybe global) of g

$\nabla f = \langle 0, 0 \rangle$ at saddle

$\lambda = 0$ ∇g anything

Level curve parallel, neither min nor max?

$g(x,y) = y - x^3 = 0$

$f = 0$

parallel neither min nor max

$\uparrow$ higher

$f(x,y) = y$

2. Suppost l, w, h denote the length, width, and height of a rectangular box.

   (a) Given $volume = V_0$ (and $V_0 > 0$), we can maximize $l + w + h$ using Lagrange multiplier. $\boxed{\text{False}}$

   (b) Given $l+w+h = N$ (and $N > 0$), we can maximize the volume using Lagrange multiplier. $\boxed{\text{True}}$

2(a) volume $= V_0 = lwh$

$l = n^2 \quad w = \frac{1}{n} \quad h = \frac{1}{n}$ (stick-like) $\qquad l = n \quad w = n \quad h = \frac{1}{n^2}$ (flatten)

$l + w + h \to \infty$ as $n \to \infty$

1. AM-GM inequality:

   Maximize $f(x_1, x_2, \cdots, x_k) = x_1 x_2 \cdots x_k$ under the constraint

$$\boxed{x_i \geq 0 \quad \forall i \qquad g = \sum_{i=1}^{k} x_i = C.}$$

$$\nabla f = \lambda \nabla g = \lambda \cdot \langle 1, 1, \cdots, 1 \rangle$$

$$f_{x_m} = \frac{\prod_{i=1}^{k} x_i}{x_m} = \lambda \quad ① \qquad \prod_{m=1}^{k} f_{x_m} = \prod_{m=1}^{k} x_m^{k-1} = \lambda^k$$

k-1 root : $\prod_{m=1}^{k} x_m = \lambda^{\frac{k}{k-1}}$ ②

① and ② $\quad x_m = \lambda^{\frac{1}{k-1}}$ all x_m equal

$$x_m = \frac{C}{k} \qquad f(\cdot-) = \frac{C^k}{k^k}$$

AM-GM

$$\frac{C}{k} = \frac{g(\cdot-)}{k} = \frac{\sum_{i=1}^{k} x_i}{k} \geq \sqrt[k]{\prod_{i=1}^{k} x_i} = \underset{\sim\sim\sim}{\sqrt[k]{f(\cdot-)}}$$ max is $\sqrt[k]{\frac{C^k}{k^k}} = \frac{C}{k}$

2. Maximize $f(x, y) = y - x^4$ along the curve $y = x^3$.

$$\nabla f = \langle -4x^3, 1 \rangle$$

$$\nabla g = \langle -3x^2, 1 \rangle \qquad \nabla f = \lambda \nabla g$$

$$-4x^3 = \lambda(-3x^2)$$

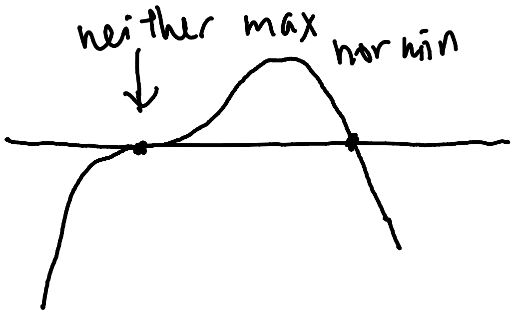
$$1 = \lambda$$

$$\rightarrow \quad -4x^3 = -3x^2 \quad \rightarrow \quad x = 0, \quad x = \tfrac{3}{4}$$

$x = 0$ maximize $x^3 - x^4 = x^3(1-x)$

root of multiplicity 3 at 0

" " " 1 at 1



$x = \frac{3}{4}$ global max

3. Let $\vec{v} = \langle a, b, c \rangle$. Maximize $|\vec{v} \cdot \langle x, y, z \rangle|$ given $x^2 + y^2 + z^2 = 1$.

$$f = |ax + by + cz|^2 = (ax + by + cz)^2$$

$$g(x, y, z) = x^2 + y^2 + z^2 \qquad g = 1$$

$$\nabla f = \langle a, b, c \rangle \cdot (2(ax + by + cz))$$

$$\nabla g = \langle 2x, 2y, 2z \rangle$$

$$\nabla f = \lambda \nabla g \ : \ \nabla f = 0 \text{ or } \langle a, b, c \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$$

$\langle x, y, z \rangle \parallel \langle a, b, c \rangle$ max

$\langle x, y, z \rangle \perp \langle a, b, c \rangle$ min