1005 - DIS 202

Monday, October 5, 2020 2:00 AM

Use Lagrange multiplier to find the point on a line closest to origin.

2D: $y = kx + b$ minimize $f(x,y) = x^2 + y^2$

$g(x,y) = y - kx = b$

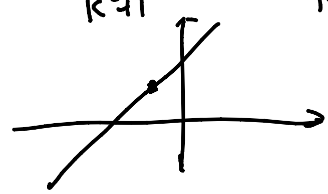
$\nabla f = \langle 2x, 2y \rangle$ $\quad \nabla g = \langle -k, 1 \rangle$

$\nabla f = \lambda \nabla g$ $\quad \left.\begin{array}{l} 2x = \lambda(-k) \\ 2y = \lambda \end{array}\right\} \; 2x = 2y(-k)$

$x = -ky = -k(kx + b)$

$(k^2+1)x = -kb$

$k = 0 \quad (0, b)$ $\qquad x = \frac{-kb}{k^2+1} \quad y = \frac{-k^2 b}{k^2+1} + b$



$k=1 \quad b=1 \qquad x = -\frac{1}{2} \qquad y = \frac{1}{2}$

3D minimizing $f(x,y,z) = x^2 + y^2 + z^2$

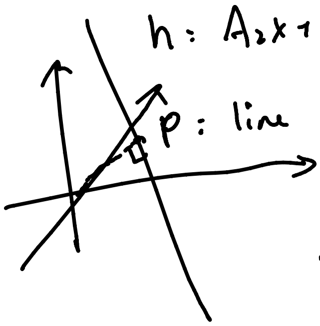
① Closest pt on a plane to origin

$g(x,y,z) = b$ tangent to the sphere

$$\begin{pmatrix} f_x \\ f_y \\ f_y \end{pmatrix} = \lambda \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix}$$ because tangent $\leftrightarrow$ surface normal parallel

② Closest pt on a line closest to origin (3D)

$\left.\begin{array}{l} g: A_1x + B_1y + C_1z = D_1 \\ h: A_2x + B_2y + C_2z = D_2 \end{array}\right\}$ intersect at line L



P: line is tangent to the sphere at P, $L \perp \nabla f$

Also, L is on $g = D_1$ & $h = D_2$

know: $L \perp \nabla g, \nabla h \rightarrow L \perp \lambda_1 \nabla g + \lambda_2 \nabla h$

As λ_1, λ_2 varies, $\lambda_1 \nabla g + \lambda_2 \nabla h$ span all directions perp to L

in particular, ∇f is within the span of $\lambda_1 \nabla g + \lambda_2 \nabla h$

(Can pick λ_1, λ_2 so $\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$)

$f \quad g_1 \; g_2 \; g_3 \cdots g_n$ just 1 eqn, use $g_i = 0 \; \forall i$

$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \cdots + \lambda_n \nabla g_n$ more

Problem #2. Find all solutions (x, y, z, λ) of the equations given by the Lagrange multiplier method for the problem of determining the points on the surface $z^2 = xy + 4$ closest to the origin.

$f(x,y,z) = x^2 + y^2 + z^2$

$g(x,y,z) = z^2 - xy = 4$

$\nabla f = (2x, 2y, 2z) \qquad \nabla g = (-y, -x, 2z)$

$\nabla f = \lambda \nabla g : \; 2z = \lambda 2z$ 3rd component

$\Rightarrow \lambda = 1$

$2x = -y$
$2y = -x$ $\quad x = y = 0 \quad g: z^2 = 4 \quad z = \pm 2$

$\boxed{(0,0,2,1) \qquad (0,0,-2,1)}$

Find the potential extrema of the function $f(x, y) = x^2 + 3xy + y^2 - x + 3y$ subject to the constraint that $0 = g(x, y) = x^2 - y^2 + 1$. * uses $g(x,y) = 0$

$\nabla f = (2x + 3y - 1, 3x + 2y + 3) \qquad \nabla g = (2x, -2y)$

$2x + 3y - 1 = \lambda 2x$ ①

$3x + 2y + 3 = \lambda(-2y)$ ②

$x^2 - y^2 + 1 = 0$ ③

$y① + x②$ $\quad 2xy + 3y^2 - y + 3x^2 + 2xy + 3x = 0$

$2xy + 3x^2 + 3 - y + 3x^2 + 2xy + 3x = 0$

Find the extrema of the function $F(x,y) = 2y + x$ subject to the constraint $0 = g(x,y) = y^2 + xy - 1$.

$\nabla F = (1,2)$ $\quad$ $\nabla g = (y, 2y+x)$

$\nabla F = \lambda \nabla g$ $\quad$ $1 = \lambda y$

$2 = \lambda(2y + x)$

$\lambda x = 0 \to$ $\quad$ $x = 0$ $\quad$ $y^2 - 1 = 0$

$y = \pm 1$

$\lambda = 0$ $\quad$ $\cancel{1 = \lambda y}$

$$\boxed{\text{max : } y = 1 \quad F = 2 \qquad \text{min } y = -1 \quad F = -2}$$

Find the extrema of $F(x,y) = x^2 y - \ln(x)$ subject to $0 = g(x,y) := 8x + 3y$.

"$F(-\frac{1}{2}, \frac{4}{3}) = \frac{1}{3} - \underline{\ln(-\frac{1}{2})}$"

$\nabla F = (2xy - \frac{1}{x}, x^2)$ $\quad$ $\nabla g = (8,3)$

$2xy - \frac{1}{x} = \lambda \cdot 8$ $\quad$ $2x^2 y - 1 = \lambda 8x$

$x^2 = \lambda \cdot 3$ $\quad$ $2yx^2 = \lambda 3 \cdot (2y) = \lambda 6y$

$\lambda 6y - \lambda 8x = 1$

$\lambda(6y - 8x) = 1$ $\quad$ $8x = -3y$

$\lambda \cdot 9y = 1$ $\quad$ $y = \frac{1}{9\lambda}$

$x = \frac{-1}{24\lambda}$

$\frac{1}{24^2 \lambda^2} = \lambda \cdot 3$ $\quad$ $\lambda^3 = \frac{1}{24^2 \cdot 3} = \frac{1}{1728} = \frac{1}{12^3}$

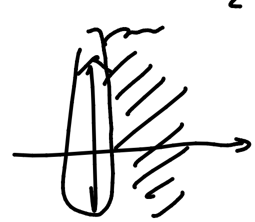
$\lambda = \frac{1}{12}$ $\quad$ $\frac{1}{\lambda} = 12$ $\quad$ $y = \frac{4}{3}$

$x = -\frac{1}{2}$

Find max/min as $x \to 0$

At any $x > 0$ still in the domain

$(0,\infty) \times (-\infty, \infty)$



Example 5.8.1.2 *Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint* $\underbrace{x^4 + y^4 + z^4 = 1}$. symmetric

$x = y = z$

$$f(x, y, z) = \underbrace{x^2 + y^2 + z^2}$$ symmetric

$x^4 = y^4 = z^4 = \left(\frac{1}{3}\right)$

$x^2 + y^2 + z^2 = \left(\frac{1}{3}\right)^{\frac{1}{2}} \cdot 3$

$= \sqrt{3}$

The later problems comes from
http://www.personal.psu.edu/sxj937/Notes/Lagrange_Multipliers.pdf
https://math.berkeley.edu/~scanlon/m16bs04/ln/16b2lec3.pdf