# 1005 - DIS 204

Monday, October 5, 2020 2:00 AM

Use Lagrange multiplier to find the point on a line closest to origin.

2D : $Ax + By = C$ minimize $x^2+y^2$

$g(x,y) = Ax + By = C$

$\nabla f = (2x, 2y) \quad \nabla g = (A, B)$

$\nabla f = \lambda \nabla g$

$$\begin{aligned} & 2x^2 = \lambda A x \\ + \; & 2y^2 = \lambda B y \\ \hline & 2x^2 + 2y^2 = \lambda(Ax+By) = \lambda C \end{aligned}$$

$$\begin{aligned} & B\,2x = \lambda A B \\ - \; & A\,2y = \lambda B A \\ \hline & 2Bx - 2Ay = 0 \end{aligned}$$

$\&\quad Ax + By = C$

$2Bx = 2Ay$

$BAx + B^2 y = CB$

$A^2 y + B^2 y = CB$

$$y = \frac{CB}{A^2+B^2}$$

$$x = \frac{CA}{A^2+B^2}$$

Solve directly $x^2+y^2$ " $A^2x^2 + A^2y^2$ " $= (Ax)^2 + A^2y^2$

$= (C - By)^2 + A^2 y^2$

$= (A^2+B^2)y^2 - 2BCy + C^2$

$$(A^2+B^2)\underline{\left(y^2 - \frac{2BC}{A^2+B^2}y + \frac{C^2}{A^2+B^2}\right)}$$

$$\left(y - \frac{BC}{A^2+B^2}\right)^2 + \text{constants}$$

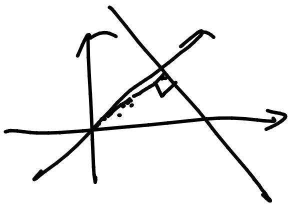
The later problems comes from
http://www.personal.psu.edu/sxj937/Notes/Lagrange_Multipliers.pdf
https://math.berkeley.edu/~scanlon/m16bs04/ln/16b2lec3.pdf

3D minimize $f(x,y,z) = x^2+y^2+z^2$ on $L$

given plane $g, h$ (intersecting at a line $L$)

$L$ tangent to the sphere (radius = distance)

at the minimizing point



$L \perp \nabla f$

$L$ on $g, h$ : $L \perp \nabla g, \nabla h \rightarrow L \perp \lambda_1 \nabla g + \lambda_2 \nabla h$

$g, h$ not parallel : $\lambda_1 \nabla g + \lambda_2 \nabla h$ covers all direction perp to $L$

$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$ (also applies $f, g_1, g_2, g_3, \dots, g_n$)

**Problem #2.** Find all solutions $(x, y, z, \lambda)$ of the equations given by the Lagrange multiplier method for the problem of determining the points on the surface $z^2 = xy + 4$ closest to the origin.

$f(x,y,z) = x^2+y^2+z^2$ (minimize)

$(0,0,2,1)$

$(0,0,-2,1)$

$g(x,y,z) = z^2 - xy$ $\quad \nabla g = (-y, -x, 2z)$

$\nabla f = (2x, 2y, 2z)$

along $y = x$ ↑

$z$ ↑

$\nabla f = \lambda \nabla g$ : 3rd component

$2z = \lambda 2z$

$(2,-2,0)$

$x = 2 \; y = -2$

$x = -2 \; y = 2$

$\leftarrow z = 0$ or $\lambda = 1 \rightarrow \left.\begin{array}{l} 2x = -y \\ 2y = -x \end{array}\right\} x = y = 0$

$xy = -4$

$2x^2 = -\lambda y x$

$2y^2 = -\lambda x y$

$x^2 - y^2 = 0$

$|x| = |y|$

$z^2 = 4 \quad z = \pm 2$

smaller : $(0, 0, \pm 2)$

Find the extrema of the function $F(x, y) = 2y + x$ subject to the constraint $0 = g(x, y) = y^2 + xy - 1$.

$\nabla F = (1, 2)$ $\quad \nabla g = (y, 2y + x)$

$1 = \lambda y$ - ①

$2 = \lambda(2y + x)$ - ②

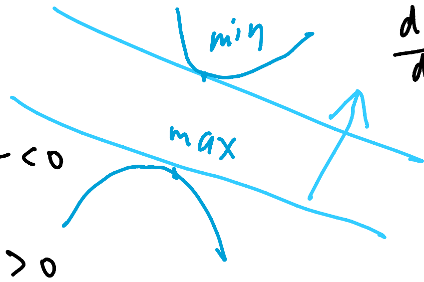
② - 2 ① $\quad \lambda x = 0 \quad \lambda = 0 \rightarrow$ ~~$1 = \lambda y$~~

$x = 0 \quad y^2 - 1 = 0 \quad y = \pm 1$

$(0, 1) \quad (0, -1)$

$y^2+xy-1=0$ $\qquad 2y\frac{dy}{dx}+y+x\frac{dy}{dx}=0$ $\qquad \frac{dy}{dx}=\frac{y}{2y+x}$

$$\frac{d^2y}{dx^2}=\frac{\frac{dy}{dx}(2y+x)-y(2\frac{dy}{dx}+1)}{(2y+x)^2}$$




$$=\frac{y-\frac{2y^2}{2y+x}-y}{(2y+x)^2}$$

$$=\frac{-2y^2}{(2y-x)^3}$$

max $\underline{x=0 \quad y=1}$ $\qquad \frac{-2}{2^3}<0$

min $\underline{x=0 \quad y=-1}$ $\qquad \frac{-2}{(-2)^3}>0$

Find the extrema of $F(x,y)=x^2y-\ln(x)$ subject to $0=g(x,y):=8x+3y$.

$\nabla F=\langle 2xy-\frac{1}{x},\ x^2\rangle$ $\qquad 2xy-\frac{1}{x}=8\lambda \cdots ①$

$\nabla g=(8,3)$ $\qquad x^2=3\lambda \cdots ②$

$x ① - 2y ②$

$g: \underline{8x=-3y}$ $\qquad \cancel{2x^2y}-1-\cancel{2yx^2}=8\lambda x-6\lambda y$

$x=\frac{-1}{24\lambda}$ ③ $\qquad =-3\lambda y-6\lambda y$

$y=\frac{1}{9\lambda}$ ④

③ → ① $\frac{1}{24^2\lambda^2}=3\lambda$ $\qquad \lambda^3=\frac{1}{12^3}$ $\qquad \lambda=\frac{1}{12}$

$\frac{1}{\lambda}=12$

$y=\frac{4}{3}$ $\quad x=-\frac{1}{2}$ → not in the domain

Domain : $x>0$ ($x=0$ is not included)

No boundary pts : No extrema.

**Example 5.8.1.2** *Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint $x^4+y^4+z^4=1$.*

$$f(x,y,z)=x^2+y^2+z^2$$ – symmetric

$x^4+y^4+z^4=1$ $\qquad \frac{1}{\lambda}\begin{pmatrix}2x\\2y\\2z\end{pmatrix}=\begin{pmatrix}4x^3\\4y^3\\4z^3\end{pmatrix}$ $\qquad |x|=|y|=|z|$

$x,y,z=\pm(\frac{1}{3})^{\frac{1}{4}}$

$\underline{\text{max.}}$

$4x^4+4y^4+4z^4=4$ $\qquad x,y,z\neq 0$

$\frac{1}{\lambda}(2x^2+2y^2+2z^2)=4$

$1=\lambda 2x^2$

$1=\lambda 2y^2$

$1=\lambda 2z^2$

$x,y,z=0$ $\qquad (0,0,1)$

$(0,0,-1)$ etc

min