Math 53 DIS 202/204 **Jiazhen Tan (tanjz@berkeley.edu)**
Multivariable Calculus *September 4, 2020*

Today's discussion. I will be in the main room for discussing questions. I will broadcast messages to the breakout rooms with the problem number we are discussing, and also update it in the shrib document, if you want to drop by for a certain problem. `https://shrib.com/#593c94-sep04`

There are two breakout rooms, one for people working on HW/practice quiz, and one for working on problems on the handout.

Problems. Those with an asterisk are harder than homework problems.

1. How many loops does $r = 1 + 5\cos 4\theta$ have?
2. What is the minimal range for θ to trace the entire $r = \sin \frac{2}{5}\theta$ curve?
3. What is the area of one loop in $r = 4\sin 3\theta$?
4. What is the 3D shape $\max\{|x|, |y|, |z|\} \leq 1$?

5*. What is the 3D shape $|x| + |y| + |z| \leq 1$?

5. What is the plane that passes through $(3, 1, 4)$, $(2, 2, 6)$, and $(-1, 3, 0)$?
6. What is the area of the parallelogram spanned by $\langle 1, 4, 5\rangle$ and $\langle 2, 1, 0\rangle$?

8*. Cross product is only defined for vectors with 3 components. Can we define an analogous operation $\boxtimes$ for vectors with 2 or 4 components, that also satisfy $|\mathbf{v} \boxtimes \mathbf{u}| = |\mathbf{v}||\mathbf{u}|\sin\theta$?

9*. The equation $x + y + z + w = 0$ defines a 3D-space inside $\mathbb{R}^4$. Since it is a 3D space, can we define a cross product for vectors $\langle x, y, z, w\rangle$ satisfying $x + y + z + w = 0$?

We want the resulting vector to point at the direction give by right-hand rule and has size equal to the parallelogram spanned by the two vectors.

Definitions FYI.

- **(Euclidean) Length of a vector.**

$$\text{Let } \mathbf{v} = \langle v_1, v_2\rangle, \text{ then } |\mathbf{v}| = \sqrt{v_1^2 + v_2^2}. \qquad \text{Let } \mathbf{v} = \langle v_1, v_2, v_3\rangle, \text{ then } |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

- **Dot product.**

$$\text{Let } \mathbf{v} = \langle v_1, v_2\rangle, \mathbf{u} = \langle u_1, u_2\rangle, \text{ then } \mathbf{v}\cdot\mathbf{u} = v_1u_1 + v_2u_2.$$

$$\text{Let } \mathbf{v} = \langle v_1, v_2, v_3\rangle, \mathbf{u} = \langle u_1, u_2, u_3\rangle, \text{ then } \mathbf{v}\cdot\mathbf{u} = v_1u_1 + v_2u_2 + v_3u_3.$$

- **Determinants.**

$$\det\begin{bmatrix} a & b \\ c & d\end{bmatrix} := \begin{vmatrix} a & b \\ c & d\end{vmatrix} = ad - bc$$

$$\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{bmatrix} := \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix} + a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1\end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix}$$

For 3×3 determinants, notice in the equation I gave, all signs are positive, and the two columns in the middle block are switched. The subscript of a, b, and c are all in the $1 \to 2 \to 3 \to 1$ order.

- **Cross Product.** For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, there are several equivalent definitions of $\mathbf{a} \times \mathbf{b}$.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{aligned}$$

The cross product $\mathbf{a} \times \mathbf{b}$ is **orthogonal** to both $\mathbf{a}$ and $\mathbf{b}$, and its direction is given by *right-hand rule*: Let right hand fingers point at $\mathbf{a}$ and bend to $\mathbf{b}$. The thumb gives the direction of $\mathbf{a} \times \mathbf{b}$.

- **Angle θ between vectors v and u.**

$$\mathbf{v} \cdot \mathbf{u} = |\mathbf{v}||\mathbf{u}| \cos\theta \quad |\mathbf{v} \times \mathbf{u}| = |\mathbf{v}||\mathbf{u}| \sin\theta$$

Two nonzero vectors are **perpendicular** if the above equations imply $\cos\theta = 0$ or $\sin\theta = \pm 1$.

- **Area and Volume.**
  - ► Let $\mathbf{v} = \langle v_1, v_2 \rangle$, $\mathbf{u} = \langle u_1, u_2 \rangle$ and $A = \begin{bmatrix} v_1 & v_2 \\ u_1 & u_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix}$.

    The parallelogram formed by $\mathbf{v}, \mathbf{u}$ has (unsigned) area $|v_1 u_2 - v_2 u_1| = |\det(A)|$.
  - ► Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$.

    The parallelogram formed by $\mathbf{v}, \mathbf{u}$ has area $\mathbf{v} \times \mathbf{u} = |\mathbf{v}||\mathbf{u}| \sin\theta$.
  - ► Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$.

    The parallelotope formed by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is defined by the set of all points

$$P = \{ r\mathbf{u} + s\mathbf{v} + t\mathbf{w} \mid r, s, t \in [0, 1] \}$$

and its (unsigned) volume is

$$V = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}, \quad \text{vol}(P) = |\det(V)| = u \cdot (v \times w)$$

- **Equation of a line.** Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. A line, in the direction of $\mathbf{a}$ and passes through $\mathbf{b}$, is given by

$$t \cdot \mathbf{a} + \mathbf{b}, t \in \mathbb{R}.$$

Its corresponding parametric equation is $x = a_1 t + b_1, y = a_2 t + b_2, z = a_3 t + b_3$.

- **Equation of a 3D plane.** The plane perpendicular to $\mathbf{v}$, with a shift of $\mathbf{a}$ from the origin, is the set

$$\{ \mathbf{w} + \mathbf{a} \mid \mathbf{v} \cdot \mathbf{w} = 0 \}.$$

Exercises with section numbers comes from ***Multivariable Calculus***, **Eighth Edition,** James Stewart.