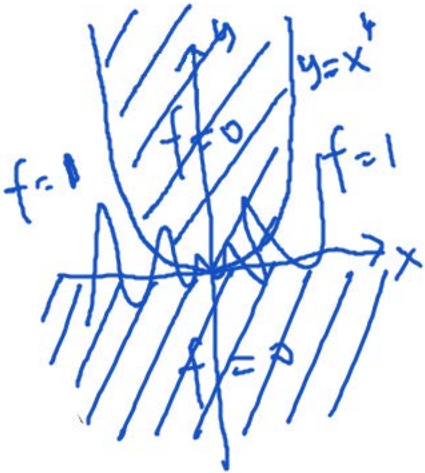
14.2.44

$y = g(x)$



1. Partial derivation

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$f(x, a, b, c)$.

$\frac{\partial f}{\partial x} = 2ax + b$

$\frac{\partial f}{\partial a} = x^2$

$\frac{\partial f}{\partial b} = x$

2. Tangent Planes

$x^2 + y^2 + z^2 = 1$

$\vec{p} : (x_0, y_0, z_0)$ on

$\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$

$Ax + By + Cz = D$

$\frac{\partial}{\partial x} Ax + By + Cz$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{-A}{C}$

$\frac{\partial z}{\partial y} = -\frac{B}{C}$

$\frac{\partial}{\partial x}(x^2 + y^2 + z^2) = \frac{\partial}{\partial x}(1)$

$2x + 2z\frac{\partial z}{\partial x} = 0$

$\frac{\partial z}{\partial x} = \frac{-x_0}{z_0}$ at $\vec{p}$

$\frac{\partial z}{\partial y} = \frac{-y_0}{z_0}$ at $\vec{p}$

$A + C\frac{\partial z}{\partial x} = 0$

$B + C\frac{\partial z}{\partial y} = 0$

$\frac{x_0}{z_0}x + \frac{y_0}{z_0}y + z = 0$

$\langle x_0, y_0, z_0 \rangle$ normal

$\frac{\partial z}{\partial x}\Big|_{\substack{\text{sphere} \\ \text{at } \vec{p}}} x + \frac{\partial z}{\partial y}\Big|_{\substack{\text{sphere} \\ \text{at } \vec{p}}} y - z = 0$

1. Partial Derivatives

$$f(x) = ce^{kx}$$

$$f'(x) = cke^{kx}$$

$$f(c, k, x)$$

$$\frac{\partial f}{\partial x} = cke^{kx}$$

$$\frac{\partial f}{\partial c} = e^{kx}$$

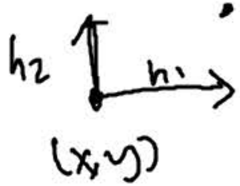
$$\frac{\partial f}{\partial k} = cxe^{kx}$$

2. Tangent Plane

$$f(x+h) \approx f(x) + f'(x) \cdot h$$

$f(x,y)$ $\quad f(x+h_1, y+h_2)$

$$\approx f(x,y) + k_1 h_1 + k_2 h_2$$



f_x, f_y exist and tangent plane DNE

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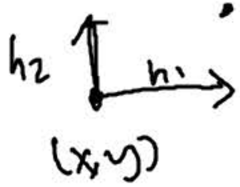

$$0 + 0 \cdot h_1 + 0 \cdot h_2$$

$$0 + \tfrac{1}{4} h_1 + \tfrac{1}{4} h_2$$

$$0 + \tfrac{1}{2} h_1$$

f_x, f_y exist and tangent plane DNE

$\rightarrow (0,0)$ along any direction

$$f(x,y) = \frac{x^2 y}{x^2+y^2} \qquad \frac{\partial f}{\partial x} = 0 \quad (y=0) \qquad h_1 = h_2$$

$$\left.\frac{\partial f}{\partial y}\right|_{x=0} = 0 \qquad f(h_1, h_1) = \frac{h_1^3}{2h_1^2} = \boxed{\tfrac{1}{2} h_1}$$

14.2 Ex 10

$$f(x,y) = \frac{5y^4 \cos^2 x}{x^4 + y^4}$$

continuous at $(0,0)$?

Limit $(x,y) \to (0,0)$ exists?

$y=0$: $f(x,0) = \frac{0}{x^4}$

$$\lim_{x \to 0} f(x,0) = 0$$

$x=0$: $f(0,y) = \frac{5y^4}{y^4} = 5$

$$\lim_{\substack{x \to 0 \\ y=0}} f(x,y) = 0$$

$$\lim_{\substack{y \to 0 \\ x=0}} f(x,y) = 0$$

$$\lim_{\substack{x=y \\ x,y \to 0}} f(x,y) = \lim_{\substack{x=y \\ y \to 0}} \frac{5y^4 \sin^2 y}{2y^4} = \lim_{y \to 0} \frac{5}{2} \sin^2 y = 0$$

consider

$x^2 + y^2 = r^2$ $r \to 0$

$$|f(x,y)| \le \frac{|5y^4 \sin^2 x|}{|x^4 + y^4|}$$ ← over

↖ under

$$|f(x,y)| \le \frac{5r^4 \cdot r^2}{\frac{r^4}{4}}$$

$\max(|x|,|y|) \ge \frac{r}{\sqrt{2}}$

$\max(x^2,y^2) \ge \frac{r^2}{2}$

$$\boxed{20r^2}$$

lim

14.2 Ex 10

$$f(x,y) = \frac{5y^4 \sin^2 x}{x^4 + y^4}$$

$y=0$ $\quad \frac{0}{x^4} \to 0$

$x=0$ $\quad \frac{0}{y^4} \to 0$

$x^2+y^2=r^2$ $\quad r \to 0$

$\max |f(x,y)| \to 0$ as $r \to 0$
on circle

$r \longleftrightarrow \delta$

point (x,y) within δ distance from $(0,0)$

if $\sqrt{x^2+y^2} \le \delta$

$|x| < \delta \quad |y| < \delta$

$$|f(x,y)| = |\sin^2 x| \cdot \left|\frac{5y^4}{x^4+y^4}\right|$$

$$\le |x|^2 \cdot \left|\frac{5y^4}{y^4}\right|$$

$$\le 5\delta^2 \to 0$$

as $\delta \to 0$

14.2 Ex 1b

$$\lim_{(x,y)\to(0,0)} \left| \frac{xy^4}{x^4+y^4} \right| \le \lim_{(x,y)\to(0,0)} \frac{|x|\cdot \cancel{y^4}}{\cancel{y^4}}$$

$|x| \to 0$

$$= \lim_{(x,y)\to(0,0)} |x|$$

$$= 0$$

Can you find a function

$$\lim_{\substack{(x,y)\to(0,0)\\ x=ky}} f(x,y) = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = 0$$

limit DNE at $(0,0)$?

$$\boxed{f(x,y) = \frac{xy^2}{x^2+y^4}}$$

$$f(0,y) = \frac{0}{0+y^4}$$

$$\frac{ky^3}{k^2x^2+y^4} = \frac{ky}{k^2+y^2}$$

$x = y^2$

$y \to 0$

$$\boxed{f(x,y) = \frac{y^4}{2y^4}} = \frac{1}{2}$$

$$f(x,y) = \frac{x^2 y}{x^2+y^2}$$

Differentiable at $(0,0)$?

"Can be approximated well by a linear thing"

Tangent plane.

f_x f_y

$$f(x+h_1, y+h_2) \approx f(x,y) + f_x \cdot h_1 + f_y \cdot h_2$$

$$f(x+h) \approx f(x) + f'(x) \cdot h$$

$f(x+h_1, y+h_2)$, h_1, h_2 small

$f_x\big|_{\text{at }(0,0)} = 0$ $f_y\big|_{\text{at }(0,0)} = 0$.

$$f(x+h_1, y+h_2) = f(x,y) + 0 \cdot h_1 + 0 \cdot h_2$$

$$f(h_1, h_1) = \frac{h_1^3}{2h_1^2} = \frac{1}{2} h_1$$

$$f(0,0) + 0h_1 + 0h_2$$

$$0 + \frac{1}{4} h_1 + \frac{1}{4} h_2$$

Could we find a function

$f(x,y)$, $\lim f(x,y) \to 0$

along any line $x = ky$ or $x = 0$

but limit DNE?

$$f(x,y) = \frac{xy^2}{x^2+y^4}$$

$x = ky$ $\quad f(ky,y) = \frac{ky^3}{k^2y^2+y^4} = \frac{ky}{k^2+y^2} \to 0$ as $y \to 0$

$$f(0,y) = \frac{0}{0+y^4} = 0 \text{ as } x \to 0$$

$x = y^2$ $\quad f(y^2,y) = \frac{1}{2}$

$$p(x,y) = Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4$$

$$\frac{p(x,y)}{x^2+y^2} = f(x,y)$$ when is $f_{xy} = f_{yx}$? (For which A B C D E)

f_x f_y $f_{xy} = (f_x)_y$

$f_{yx} = (f_y)_x$

$$f(x,y) = \frac{x^4}{x^2+y^2}$$

$$f_x = \frac{x^3(y^2-x^2)}{(x^2+y^2)^2}$$

$$f_{xy} = \frac{(x^2+y^2)^2(3x^2(y^2-x^2)+x^3(-2x)) - 4x(x^2+y^2)\,x^3(y^2-x^2)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(3(y^2-x^2)+(-2x^2)) - 4x^2(y^2-x^2)}{(x^2+y^2)^3}\, x^2$$

f_y f_{yx}

$f_{xy} = f_{yx}$ iff $B = D$