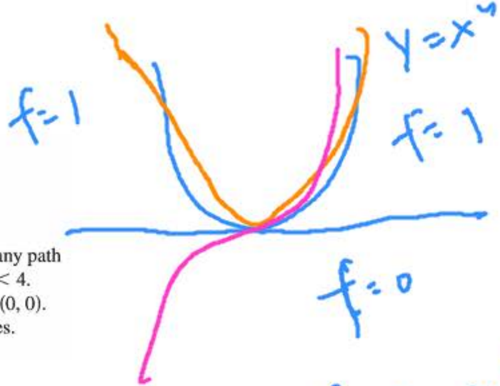
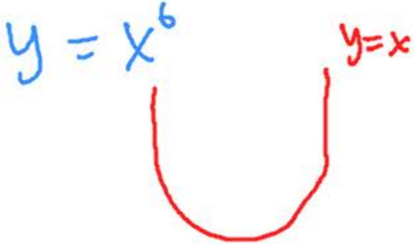
44. Let

$$f(x, y) = \begin{cases} 0 & \text{if } y \leqslant 0 \quad \text{or} \quad y \geqslant x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

(a) Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along any path through $(0, 0)$ of the form $y = mx^a$ with $0 < a < 4$.
(b) Despite part (a), show that f is discontinuous at $(0, 0)$.
(c) Show that f is discontinuous on two entire curves.



$y \leq 1$



$y = mx^a > x^4$ ⇐ solve for this

$a = 1, 3 \qquad x \geq 0$

$a = 0, 2 \qquad$ all x around 0

$f(x,y) = 0$ as $(x,y) \to (0,0)$ along these

- $y = 10x \qquad |x| < 1$
- $y = x^2$
- $y = 3x^3 \qquad 0 \leq |x| < 3 \qquad$ when $x < 0$, $y = 3x^3 < 0$

$f(x) = (x+3,\ x^2-2x+1,\ x^3-2x^2-\cancel{x}-6)$

when is tangent line passing through origin?

$\frac{df}{dx} = (1,\ 2x-2,\ 3x^2-4x-1)$

$x = x_0$

tgt line $g(t) = (x_0+3,\ x_0^2-2x_0+1,\ x_0^3-2x_0^2-x_0-6) - t(1,\ 2x_0-2,\ 3x_0^2-4x_0-1)$

$= (0,0,0)$

$x_0+3 = t$

$x_0^2-2x_0+1 = t(2x_0-2)$

$x_0^3-2x_0^2-x_0-6 = t(3x_0^2-4x_0-1)$

$x_0^2+6x_0-7=0$

$(x_0-1)(x_0+7)=0$

$x_0 = 1$

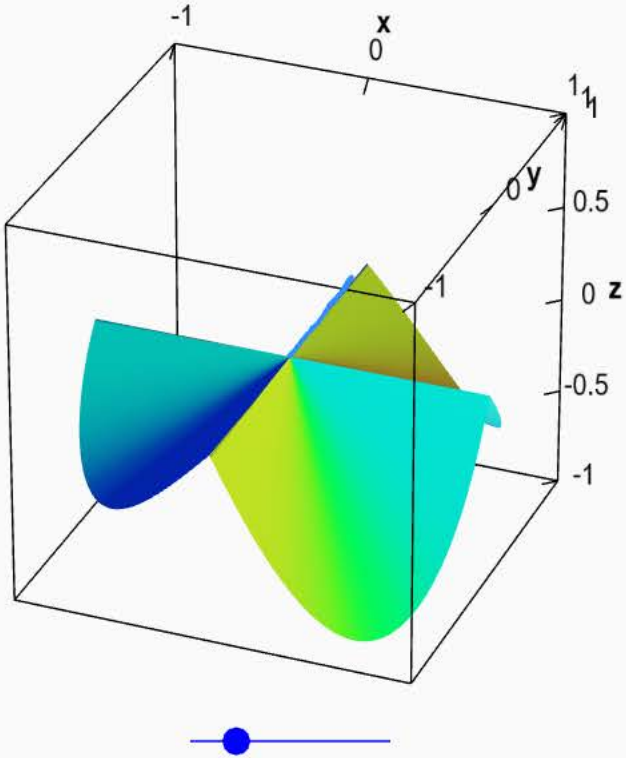
$2x_0^3+7x_0^2-12x_0+3=0$

$(x_0-1)(2x_0^2+9x_0-3)$

no real roots

f_x, f_y exist at (a,b)

must f be differentiable at (a,b)



f_x, f_y exists & both 0 at $(0,0)$

$f(x,y) \approx f(0,0) + ax + by$

f_x, f_y exist at (p,q)

$\not\Rightarrow$ f is continuous at (p,q)

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$x=0$ or $y=0$

$f(x,y) = 0$

$x = y$

$f(x,y) = \frac{1}{2}$

$\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 0$

$\sin 2\theta = 0$

$f(x,y) = g(r,\theta) = 0$

$x = 0 \quad y \to 0$

$x \to 0 \quad y = 0$

polar coord. $x = r\cos\theta \quad y = r\sin\theta$

$$f(x,y) = g(r,\theta) = \frac{r\cos\theta \, r\sin\theta}{r^2} = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$$f(x,y) = \frac{xy}{x^2+y^2}$$

f_x f_y both exist at $(0,0)$

$x=0$ $y \to 0$ $f(x,y)=0$

$x \to 0$ $y=0$ $f(x,y)=0$

$x=y \to \frac{1}{2}$

$\not\Rightarrow$ $f(x,y)$ continuous

differentiable $\Longrightarrow$ continuous

f_x f_y exist does not imply differentiable

polar coordinates

$f(x,y) = g(r,\theta)$

$x = r\cos\theta$

$y = r\sin\theta$

$$g(r,\theta) = \frac{r^2\cos\theta\sin\theta}{r^2(\cos^2\theta+\sin^2\theta)}$$

$r \neq 0$ $= \cos\theta\sin\theta$

$= \frac{1}{2}\sin 2\theta$

$r \to 0$ $\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 0$

$\sin 2\theta = 0 \Rightarrow g(r,\theta) \to 0$

θ anything else $g(r,\theta) \to \frac{1}{2}\sin 2\theta$

4. when tangent plane horizontal

① $f(x,y) = 4x^2 + y^2$

② $f(x,y) = xy + y^3$

Polynomial in n variables then is differentiable $\mathbb{R}^n \to \mathbb{R}$

— polynomial in x, y $\mathbb{R}^2 \to \mathbb{R}$

③ $f(x,y) = \sqrt{x} + \sqrt{y}$ $\Rightarrow$ (a,b) $f_x(a,b)$ $f_y(a,b)$

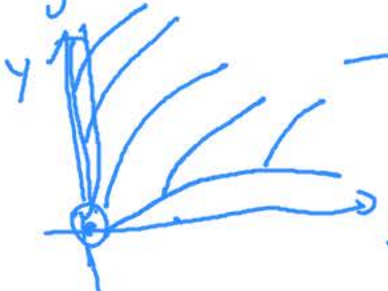
$(x, y > 0)$

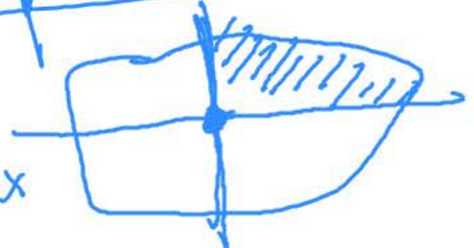
$(a+h, b+k) = f(a,b) + f_x \cdot h + f_y \cdot k$

* Do they have tangent planes in all of domain.

$y = |x|$

$g(x) = \sqrt{x}$





$f_x = y \quad f_y = x + 3y^2$

$x = 1,\ y = 1$

$f(x,y) = xy + y^3$

surface in $\mathbb{R}^3$

$L(x+h, y+k) \simeq f(x,y) + h \cdot f_x + k f_y + h^a k^b \ldots$ $\quad a+b \geq 2$

(x, y) fixed $\quad f_x, f_y$ const.

$L(1+h, 1+k) = 2 + h + 4k$

$(1+h,\ 1+k,\ 2+h+4k)$ $\quad h \in \mathbb{R}$, $k \in \mathbb{R}$