Evans Fall 2010 Midterm 1:

Monday, September 28, 2020 10:10 AM

Problem #1. (a) Compute ∇f for

$$f(x,y) = e^{x^2y+\sin(xy)}$$

(b) Compute ∇g for

$$g(x,y,z) = (x^2+y^3+z^4)^{-1}$$

(a) $\vec{\nabla} f = \langle f_x, f_y \rangle = \langle (2xy + y\cos x)\, e^{x^2y+\sin(xy)},$

$(x^2 + x\cos y)\, e^{x^2y+\sin(xy)} \rangle$

(b) $\vec{\nabla} g = \langle g_x, g_y, g_z \rangle = -\langle 2x, 3y^2, 4z^3 \rangle \cdot (x^2+y^3+z^4)^{-2}$

Problem #2. Find the critical points of the function

$$f(x,y) = x^4 + 2y^2 - 4xy$$

and classify each as a local maximum, local minimum or saddle point.

$f_x = 4x^3 - 4y = 0 \Rightarrow y = x^3$

$f_y = 4y - 4x = 0 \Rightarrow y = x$

$(-1,-1) \quad (0,0) \quad (1,1)$

$f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 4$

$D(-1,-1) = f_{xx} f_{yy} - f_{xy}^2$ $\quad f_{xx} > 0$ minimum

$= 12 \cdot 4 - 4^2 > 0$

$D(0,0) = 0 \cdot 4 - 4^2 < 0$ saddle

$D(1,1) = 12 \cdot 4 - 4^2 > 0$ $\quad f_{xx} > 0$ minimum

$$\boxed{\text{Critical pts} \quad (-1,-1),\ (1,1) \text{ local minimum} \quad (0,0) \text{ saddle pt}}$$

Problem #3. The position vector $\mathbf{r}(t)$ of a particle moving in three dimensions satisfies

$$\mathbf{r}' = \mathbf{r} \times \mathbf{a}$$

where $\mathbf{a}$ is a fixed vector.

Show that either the particle is not moving or else its motion lies within a circle.

(Hint: Show $|\mathbf{r}|$ and $\mathbf{r} \cdot \mathbf{a}$ are constant.)

continuous, derivative = 0 everywhere

Notice $|r|$, $r \cdot a$ only depend on t

has only one output

$$\underset{\text{const}}{r} \cdot r = \underset{\text{const}}{|r|^2}$$

$\mathbb{R} \to \mathbb{R}$ Fund thm of Calc:

$F'(t) = 0 \;\; \forall t \Rightarrow F(t)$ const.

$\cancel{|r|}$ $r \cdot r$, $r \cdot a$ $\qquad r = \langle f(t), g(t), h(t) \rangle$

$$\left.\begin{aligned} (r \cdot r)' = \left(f(t)^2 + g(t)^2 + h(t)^2 \right)' &= 2f'f + 2g'g + 2h'h \\ &= 2r'(t) \cdot r(t) \\ &= 2(r \times a) \cdot r = 0 \end{aligned}\right\} \text{a sphere of radius } r$$

$a = (a_1, a_2, a_3)$

$$(r \cdot a)' = (a_1 f + a_2 g + a_3 h)' = (a_1 f' + a_2 g' + a_3 h')$$
$$= a \cdot r' = a \cdot (r \times a) = 0$$

$r \cdot a$ constant $\Rightarrow$ in the plane $a_1 x + a_2 y + a_3 z = C$

plane & sphere, nonempty intersection $(C = r(0) \cdot a)$

must intersect on a circle or a point

Problem #4. Find the area of the region inside the curve

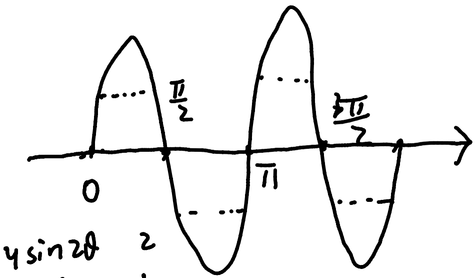
$$r = 4\sin 2\theta$$

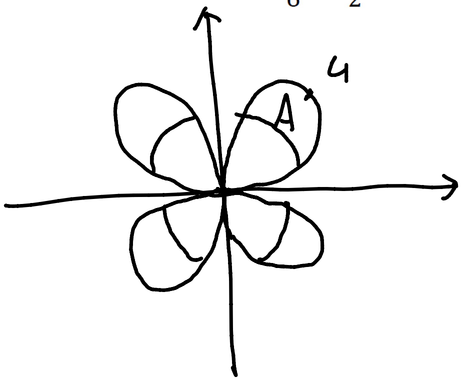
and outside the circle

$$r = 2$$

for $0 \le \theta \le \frac{\pi}{2}$.

(Reminders: $\sin\frac{\pi}{6} = \frac{1}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$)





$4\sin 2\theta \quad 2$

$$A = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{1}{2}\left(R^2 - r^2\right) d\theta$$

$$= \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{1}{2}\left((4\sin 2\theta)^2 - 4\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 8(1-\cos 4\theta)\, d\theta - \int_{\frac{\pi}{2}}^{\frac{5\pi}{12}} 2\, d\theta$$

$$= \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} -4\cos 4\theta\, d\theta + \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 2\, d\theta$$

$$= \Big[-\sin 4\theta\Big]_{\frac{\pi}{12}}^{\frac{5\pi}{12}} + \frac{2}{3}\pi$$

$$= -\sin\frac{5\pi}{3} + \sin\frac{\pi}{3} + \frac{2}{3}\pi$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{2}{3}\pi = \boxed{\sqrt{3} + \frac{2}{3}\pi}$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$$
$$= 1 - 2\sin^2\varphi$$
$$2\sin^2\varphi = 1 - \cos 2\varphi$$

Problem #5. Assume that the two equations

$$f(x,y,z) = 0, g(x,y,z) = 0$$

together implicitly (define y as a function of x and z as a function of x.) $\vec{\nabla} f \neq 0 \quad \vec{\nabla} g \neq 0$

Find formulas for $y' = \frac{dy}{dx}$ and $z' = \frac{dz}{dx}$ in terms of the partial derivatives of f and g.

$f(x,y,z) = 0 \qquad g(x,y,z)$

$\nabla f \cdot \vec{u} = 0 \qquad \nabla g \cdot \vec{u} = 0$

$\vec{u} = \langle \Delta x, \Delta y, \Delta z \rangle$

$= \langle \Delta x, \frac{dy}{dx} \Delta x, \frac{dz}{dx} \Delta x \rangle$

$= \delta x \cdot \langle 1, y', z' \rangle$

$\langle f_x, f_y, f_z \rangle \cdot \langle 1, y', z' \rangle = 0$

$\langle g_x, g_y, g_z \rangle \cdot \langle 1, y', z' \rangle = 0$

$\vec{\nabla} f \times \vec{\nabla} g \parallel \langle 1, y', z' \rangle$

$$\boxed{y' = \frac{f_z g_x - f_x g_z}{f_y g_z - f_z g_y} \qquad z' = \frac{f_x g_y - f_y g_x}{f_y g_z - f_z g_y}}$$