Neu Fall 2009

Monday, September 28, 2020 10:04 AM

Neu FA2009 Midterm 2:

1. z is a function of x and y defined implicitly by $\left(\sin x \sin y \sin z = \frac{1}{4}\right)$. Compute $z_x\left(\frac{\pi}{4}, \frac{\pi}{6}\right)$ and $z_y\left(\frac{\pi}{4}, \frac{\pi}{6}\right)$ assuming $z\left(\frac{\pi}{4}, \frac{\pi}{6}\right) = \frac{\pi}{4}$.

$$\frac{\partial}{\partial x}(\sin x \sin y \sin z) = \frac{\partial}{\partial x}\left(\frac{1}{4}\right) \qquad \left(\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{4}\right)$$

1 $\cos x \cancel{\sin y} \sin z + \sin x \cancel{\sin y} \cos z \dfrac{\partial z}{\partial x} = 0$

2 $\cancel{\sin x} \cos y \cancel{\sin z} + \cancel{\sin x} \sin y \cancel{\cos z} \dfrac{\partial z}{\partial y} = 0$

$$\frac{\sqrt{2}}{2} \cdot \frac{1}{2} \cdot \underset{\downarrow \atop \frac{\sqrt{2}}{2}}{\sin z} = \frac{1}{4}$$

1 $\dfrac{\sqrt{2}}{2} \cdot \dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} \cdot \dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{2} \dfrac{\partial z}{\partial x} = 0 \qquad \dfrac{\partial z}{\partial x} = -1$

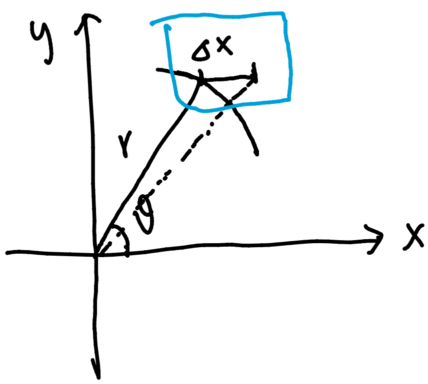
2 $\dfrac{\sqrt{3}}{2} + \dfrac{1}{2} \dfrac{\partial z}{\partial y} = 0 \qquad \dfrac{\partial z}{\partial y} = -\sqrt{3}$

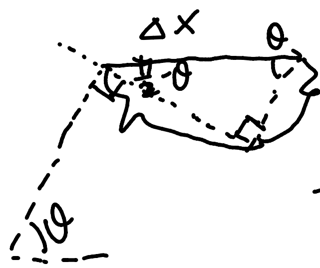
2. Given $F(x, y)$ define $f(r, \theta)$ by

$f(r, \theta) := F(r\cos\theta, r\sin\theta)$.

Suppose $f_r(\sqrt{2}, \frac{\pi}{4}) = 1$, $f_\theta(\sqrt{2}, \frac{\pi}{4}) = -1$.

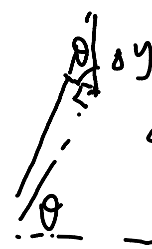
What are $F_x(1, 1)$ and $F_y(1, 1)$?





$$\Delta x\cos\theta = \Delta r$$

$$\frac{\Delta x\sin\theta}{r} = -\Delta\theta$$



$$\Delta y\sin\theta = \Delta r$$

$$-\frac{\Delta y\cos\theta}{r} = -\Delta\theta$$

$$\Delta x \cdot F_x(1,1) \approx F(1+\Delta x, 1) - F(1,1) \approx f_r(\sqrt{2}, \tfrac{\pi}{4}) \cdot (\Delta x\cos\theta) + f_\theta(\sqrt{2}, \tfrac{\pi}{4})\left(-\frac{\Delta x\sin\theta}{r}\right)$$

$$F_x(1,1) = f_r(\sqrt{2}, \tfrac{\pi}{4}) \underbrace{\cancel{\cos\theta}}_{\frac{\sqrt{2}}{2}} + f_\theta(\sqrt{2}, \tfrac{\pi}{4}) \cdot \cancel{\frac{\sin\theta}{r}} \ \frac{1}{2}$$

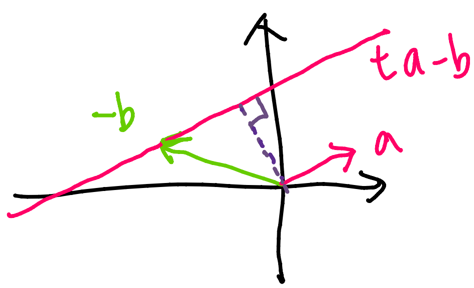
$$\boxed{= \frac{\sqrt{2}}{2} - \frac{1}{2}}$$

$$F_y(1,1) = f_r(\sqrt{2}, \tfrac{\pi}{4}) \sin\theta + f_\theta(\sqrt{2}, \tfrac{\pi}{4}) \frac{\cos\theta}{r}$$

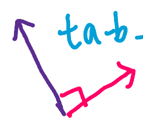
$$\boxed{= \frac{\sqrt{2}}{2} - \frac{1}{2}}$$

Neu FA2009 Midterm 1:

3. The parametric line $\underline{r}(t) = t\underline{a} - \underline{b}$ represents displacement of a particle from the origin. $\underline{a}$, $\underline{b}$ are given vectors, and $\underline{a} \neq 0$. What is the smallest distance of the particle from the origin?



At minimum distance:

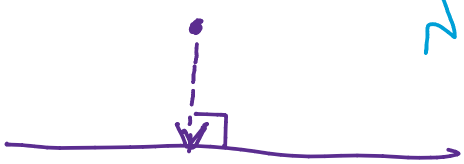


$$(t\vec{a} - \vec{b}) \cdot \vec{a} = 0$$

$$t\,\vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{a}$$

$$t = \frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}$$

find $|ta-b|$:



$ta-b,\ t \in \mathbb{R}$: arbitrary pt on the line

solving for $ta-b$ smallest :

$$\sqrt{|ta-b|^2} = (ta-b)\cdot(ta-b)$$

$$= t^2\, a\cdot a - 2t\, a\cdot b + b\cdot b$$

$$= \frac{(b\cdot a)^2}{(a\cdot a)} - 2\,\frac{(b\cdot a)^2}{(a\cdot a)} + b\cdot b = \sqrt{b\cdot b - \frac{(b\cdot a)^2}{a\cdot a}}$$

5. Find the vector function $\underline{r}(t)$ that represent uniform circular motion, of period 2π, centered about origin, with axis of rotation $\underline{w} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, and $\underline{r}(0) = (1, -1, 1)$

Suppose the radius of rotation is R.

On a plane around z:

$$(R\cos\theta, R\sin\theta, 0)$$

$$= R\cos\theta \cdot \langle 1,0,0\rangle + R\sin\theta \cdot \langle 0,1,0\rangle + 0 \cdot \langle 0,0,1\rangle$$

rotating around $\vec{w}$: replace $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$, $\langle 0,0,1\rangle$ by a triple of orthogonal vectors $\vec{x}, \vec{y}, \vec{w}$

$$r(\theta) = R\cos(\theta+\varphi)\vec{x} + R\sin(\theta+\varphi)\vec{y}$$

$$r(0) = R\cos\varphi \left\langle \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle + R\sin\varphi \left\langle \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle$$

$$= \langle 1,-1,1\rangle$$

$\sin\varphi = -1$, $\cos\varphi = 0$, $\varphi = \frac{3\pi}{2}$

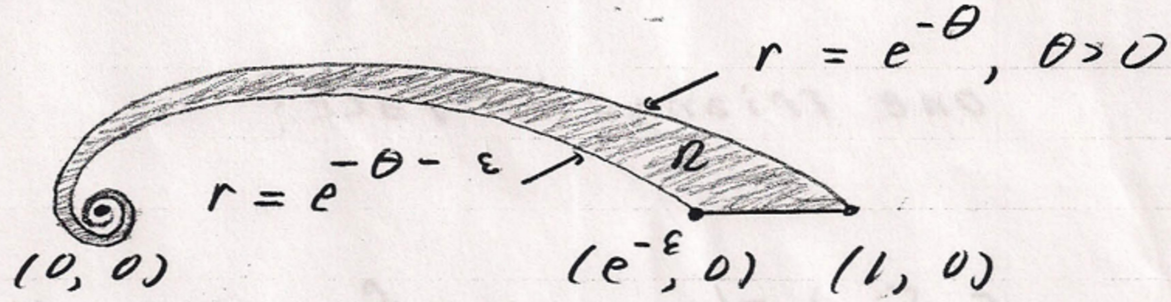
$$r(\theta) = R\cos\left(\theta + \frac{3\pi}{2}\right)\vec{x} + R\sin\left(\theta + \frac{3\pi}{2}\right)\vec{y}$$

alternatively, $x = \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ $\quad y = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle$

$$r(\theta) = R\cos\theta\,\vec{x} + R\sin\theta\,\vec{y}, \quad R = \sqrt{3}$$

$$= \left\langle \cos\theta + \sqrt{\tfrac{3}{2}}\sin\theta, -\cos\theta, \cos\theta - \sqrt{\tfrac{3}{2}}\sin\theta \right\rangle$$

2. Find the area of the region R:



ε is a positive constant.

$$A = \int_0^\infty \frac{1}{2}\left(\left(e^{-\theta}\right)^2 - \left(e^{-\theta-\varepsilon}\right)^2 \right) d\theta$$