Equidistribution Review 3

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Intervals and Cubes

- An **open interval** $I = (a, b) \subset \mathbb{R}$ is the subset

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$

and we define its length (i.e. 1-dim volume) as

$$\text{vol}(I) = b - a.$$

- A **closed interval** $I = [a, b] \subset \mathbb{R}$ is the subset

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

and we define its length as $\text{vol}(I) = b - a$.

Intervals and Cubes

- An **open cube** $Q \subset \mathbb{R}^n$ is a Cartesian product

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$$

of open intervals, and we define its volume as

$$\text{vol}(Q) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

- A **closed cube** $Q \subset \mathbb{R}^n$ is a Cartesian product

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of closed intervals, and we define its volume as

$$\text{vol}(Q) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

Note: It's really a n-dim rectangle.

Shifting by Constant

- For a set $A \subset \mathbb{R}$ and a real number $r \in \mathbb{R}$, we define

$$A + r := \{x + r \mid x \in A\}$$

- For a set $A \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$, we define

$$A + p := \{x + p \mid x \in A\}$$

Here we add the n-tuples component-by-component.

Fact: For a (open or closed) cube $A \subset \mathbb{R}^n$,

$$\text{vol}(A) = \text{vol}(A) + r \quad \forall r \in \mathbb{R}.$$

Proof: $$(b + r) - (a + r) = b - a, \quad \forall a, b, r \in \mathbb{R}$$

Set Addition and Union

- For sets $A, B \subset \mathbb{R}^n$, we define

$$A + B := \{x + y \mid x \in A, y \in B\}$$

- For disjoint sets $A, B \subset \mathbb{R}^n$, we define their disjoint union as

$$A \sqcup B := \{x \mid x \in A \text{ or } x \in B\}$$

Remark. It is almost never the case that

$$\mathrm{vol}(A + B) = \mathrm{vol}(A) + \mathrm{vol}(B) \quad \text{(even in an intuitive sense of volume).}$$

However, for cubes A and B, we define

$$\mathrm{vol}(A \sqcup B) = \mathrm{vol}(A) + \mathrm{vol}(B)$$

and more generally, $\mathrm{vol}(A \cup B) \leq \mathrm{vol}(A) + \mathrm{vol}(B)$.

Measure Zero

Definition

A set $S \in \mathbb{R}$ has measure 0 if for any $\epsilon > 0$, there exists a finite or countable collection of open intervals (I_i) such that

$$\sum \text{vol}(I_i) \leq \epsilon \quad \text{and} \quad S \subset \bigcup I_i$$

Definition

A set $S \in \mathbb{R}^n$ has measure 0 if for any $\epsilon > 0$, there exists a finite or countable collection of open cubes Q_i such that

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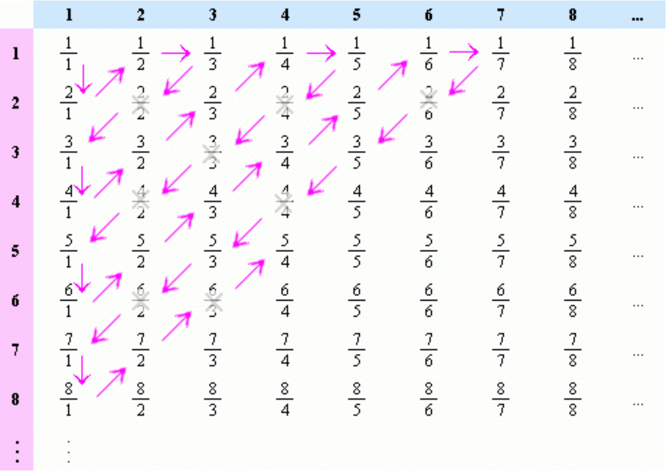
Countable Set

- Look up numberphile video "infinity is bigger than you think."
- Means you can list them out.
- $\mathbb{N}$ is countable: $1, 2, 3, 4 \ldots$
- $\mathbb{Z}$ is countable: $0, 1, -1, 2, -2, 3, -3, \ldots$
- $\mathbb{R}$ is not countable (Cantor diagonal argument)

$\mathbb{Q}$ is countable



(from `https://math.stackexchange.com/questions/501782/is-the-infinite-table-argument-for-the-countability-of-q-u`

Countable set has measure 0

- Let $E = \{e_1, e_2, \ldots,\}$ be a countable set, $\epsilon > 0$.
- Then E can be covered with

$$F = \bigcup_{i=1}^{\infty} \left(e_i - \frac{\epsilon}{2^{i+1}}, e_i + \frac{\epsilon}{2^{i+1}} \right)$$

$$\text{total length of intervals in } F = \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$$

- Hence $\mathbb{Q}$ is measure 0.

$[0, 1]$ is not measure 0

- Suppose I_i is a collection of countably many intervals that cover $[0, 1]$.
- Because $[0, 1]$ is compact, I_i has a finite subcover: i.e. there exists a finite subcollection $(I_{n_k})_{k=1}^{L}$ such that

$$[0, 1] \subseteq \bigcup_{k=1}^{L} I_{n_k}.$$

- A finite collection of intervals that cover $[0, 1]$ must have total length greater than or equal to 1.

Open, Closed, Half-Open intervals

Theorem

In the definition of measure 0, you can replace "open" with "closed" or "half-open."

Half open intervals are intervals with one side open the other side closed (e.g. $[a, b)$).

sketch.

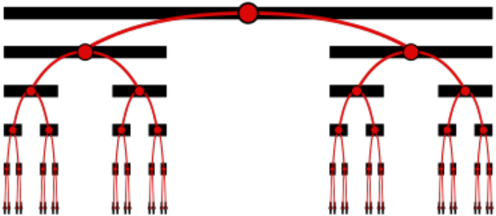
Given a collection $I_i = \bigcup_{i=1}^{\infty}(a_i, b_i)$ with total length ϵ, can take

$$I_i = \bigcup_{i=1}^{\infty}[a_i, b_i].$$

Conversely, given $I_i = \bigcup_{i=1}^{\infty}[a_i, b_i]$, can take $I_i = \bigcup_{i=1}^{\infty}(a_i - \epsilon/2^i, b_i + \epsilon/2^i)$. That has total length 2ϵ. Similar proof works for equivalence of "open" and "half-open." □

Cantor Set

- Take $[0, 1]$.
- Take away middle third $(1/3, 2/3)$. End up with $[0, 1/3] \cup [2/3, 1]$.
- Repeat for each of those two intervals.



(Taken from Wikipedia)

- Let $\mathcal{C}$ be the Cantor set, the "limit" when we repeat the construction infinitely many times.

Cantor Set

Formally, the Cantor set can be defined by:

$$C_0 := [0,1], \quad C_n := \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right) \forall n \in \mathbb{Z}^+$$

$$\mathcal{C} := \bigcap_{i=0}^{\infty} C_i$$

- C_n are all closed - a union of 2^n closed intervals.
- C_n are nested, i.e. $C_n \subset C_{n-1}, \quad \forall n \in \mathbb{Z}^+$.

True for $n = 1$; by WOP, smallest bad $n \geq 2$. Since $C_{n-1} \subset C_{n-2}$,

$$\frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right) \subset \frac{C_{n-2}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-2}}{3}\right).$$

Facts abour the Cantor set

- Cantor set is uncountable.
- Cantor set is totally disconnected.
- Cantor set is nowhere dense.
- Cantor set is closed and bounded, thus compact by Heine-Borel.
- Cantor set is a perfect set, where every point is an accumulation point (for $x \in \mathcal{C}$, points in $\mathcal{C} \setminus \{x\}$ approximate x arbitrarily well).
- Cantor set with the Euclidean metric is homeomorphic to $\{0,1\}^{\mathbb{N}}$ with the metric

$$d(\vec{x},\vec{y}) = \sum_{i \in \mathbb{N}} \frac{|x_i - y_i|}{2^i}, \quad \vec{x},\vec{y} \in \{0,1\}^{\mathbb{N}}$$

Cantor Set has measure 0

- Note that in our construction each C_n is a disjoint union of closed intervals, whose volume is the sum of their length.
- For union of disjoint intervals, we write vol as a shorthand for sum of length.

$$
\begin{aligned}
\mathrm{vol}(C_n) &= \mathrm{vol}\left(\frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)\right) \\
&\leq \frac{1}{3}\,\mathrm{vol}(C_{n-1}) + \frac{1}{3}\,\mathrm{vol}(C_{n-1}) \\
&= \frac{2}{3}\,\mathrm{vol}(C_{n-1})
\end{aligned}
$$

- Since $\mathrm{vol}([0,1]) = 1$, $\mathrm{vol}(C_n) = \left(\frac{2}{3}\right)^n \to 0$ as $n \to \infty$.

More Properties of Volume

- For a cube $Q \in \mathbb{R}^n$, if we define mQ for $m \in \mathbb{R}$ as

  $$mQ := \{m \cdot \vec{x} \mid \vec{x} \in Q\},$$

  a dilation of Q by a factor of m, and

  $$\operatorname{vol}(mQ) = |m|^n \operatorname{vol}(Q) \tag{1}$$

- In general we want equation (1) true for all nice subsets of $\mathbb{R}^n$. This inspires a definition of dimension.
- The Cantor set $\mathcal{C}$ would satisfy (if we can make $\operatorname{vol}(\mathcal{C}) \neq 0$)

  $$\operatorname{vol}(3\mathcal{C}) = 2 \operatorname{vol}(\mathcal{C})$$

  because dilating by $m = 3$ adds another copy of it in $[2, 3]$.
- What could the "dimension" of $\mathcal{C}$ be?