Equidistribution Review 5

James Leng and Jiazhen Tan

Ross

July 15, 2020

The n-dimensional Torus

- When we take $[0, 1]$ and define an equivalence relation by

$$a \sim a \ \forall a, \quad 0 \sim 1,$$

  we glued its two ends and get a circle.
- Equivalently, we can think of the object above as $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, where

$$a \sim b \in \mathbb{R} \iff a - b \in \mathbb{Z}.$$

- We can generalize this to n-tuples: $\mathbb{R}^n/\mathbb{Z}^n$ is the space where

$$a \sim b \in \mathbb{R}^n \iff a - b \in \mathbb{Z}^n.$$

- We call $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the n dimensional torus.

Structure of the n-dimensional Torus

- You can make $\mathbb{T}^n$ a group under addition, adding component by component and moding out by $\mathbb{Z}^n$.
- By-component multiplication is not well defined.

The equivalence relation works nicely with the group structure:

- If you think about $\mathbb{R}^n$ and $\mathbb{Z}^n$ as groups under addition, and consider equivalence relation making $\mathbb{R}^n$ into $\mathbb{T}^n$ as a map

$$f : \mathbb{R}^n \to \mathbb{T}^n,$$

  then

$$f(a + b) = f(a) + f(b).$$

Besides, applying f to the entire $\mathbb{R}^n/\mathbb{Z}^n$ is the same as applying it to each component.

Every dense sequence has a UD subsequence

We construct $\{y_n\}$ from $\{x_n\}$ as follows:

- Take a U.D. sequence $\{r_k\}$. For example, the sequence you get by keeping track of inserted terms of the Farey sequence.
- Pick
$$x_{n_k} \in B_{\frac{1}{2^k}}(r_k),$$
and let $y_k = x_{n_k}$.
- For N much larger than 2^m, chance of hitting an interval (a, b) is bounded above and below by Farey sequence hitting the interval
$$U_m = (a - \frac{1}{2^m}, b + \frac{1}{2^m}), \text{ resp. } L_m(a + \frac{1}{2^m}, b - \frac{1}{2^m})$$
- We can make m large, and use
$$\lim_{N\to\infty} \frac{\#|r^k \in U_m|}{N} \to |U_m|, \text{ resp. } L_m.$$

Every dense sequence has a UD rearrangement

Ideas in constructing $\{y_n\}$ from $\{x_n\}$:

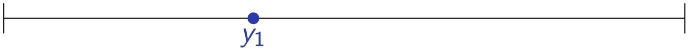
1. You want to range through all of $(0, 1)$ infinitely many times - meaning, for every interval (a, b), you should draw things from it infinitely many times in our rearrangement.
2. Be careful of the amount of oscillation - you need the interval between drawing two numbers from (a, b) small compared to the number of already drawn terms.
   As an example for the "oscillation" phenomenon in a sequence that would have seemed uniform: if you write all digits in the natural numbers in a row from the smallest to largest, the chances of 1 being the leading digit ranges from $\approx 30\%$ to $\approx 11\%$. See Benford's law.

Every dense sequence has a UD rearrangement

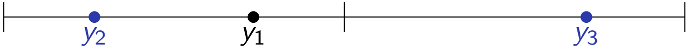
One example this construction process of $\{y_n\}$ from $\{x_n\}$:

1. Let the m^{th} triangular number $T(m) = \frac{m(m+1)}{2}$
2. Pick y_k, where
$$T(m-1)+1 \leq k \leq T(m),$$
in the interval
$$\left(\frac{k - T(m-1) - 1}{m}, \frac{k - T(m-1)}{m}\right).$$
3. The number of intervals of length $\frac{1}{m} \subset (a, b)$ goes to $(b-a)m$ as m gets large.
4. The number of terms in each iteration of step 2 is small, i.e. $O(n)$, compared to terms we already- https://www.overleaf.com/project/5f0f3f5f7b2f95000108aa51 picked out, i.e. $O(n^2)$, so the oscillation for different truncations N is also small.

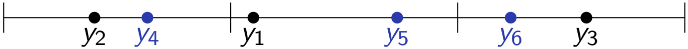
Every dense sequence has a UD rearrangement



Pick the first term within $(0, 1)$



Pick the next two terms in $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$



Pick the next three terms in $(0, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$, $(\frac{2}{3}, 1)$