Equidistribution Review 6

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July 18, 2020

Measure Zero

Definition

A set $S \in \mathbb{R}$ has measure 0 if for any $\epsilon > 0$, there exists a finite or countable collection of open intervals (I_i) such that

$$\sum \text{vol}(I_i) \leq \epsilon \quad \text{and} \quad S \subset \bigcup I_i$$

Definition

A set $S \in \mathbb{R}^n$ has measure 0 if for any $\epsilon > 0$, there exists a finite or countable collection of open cubes Q_i such that

$$\sum \text{vol}(Q_i) \leq \epsilon \quad \text{and} \quad S \subset \bigcup Q_i$$

Terminology

When talking about measure 0, use “almost everywhere.” For example, if $f(x) = g(x)$ almost everywhere, that means that the set for which $f(x) \neq g(x)$ is measure 0. If $f, g : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

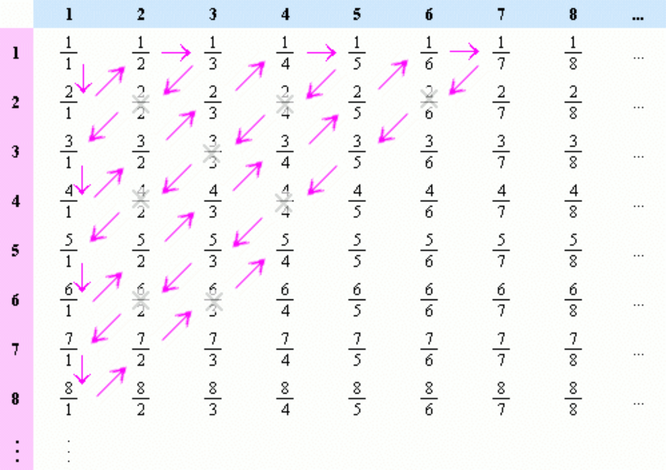
$$g(x) = 0$$

are equal almost everywhere.

Countable Set

- Look up numberphile video “infinity is bigger than you think."
- Means you can list them out.
- $\mathbb{N}$ is countable: $1, 2, 3, 4 \ldots$
- $\mathbb{Z}$ is countable: $0, 1, -1, 2, -2, 3, -3, \ldots$
- $\mathbb{R}$ is not countable (Cantor diagonal argument)

$\mathbb{Q}$ is countable



(from `https://math.stackexchange.com/questions/501782/is-the-infinite-table-argument-for-the-countability-of-q-u`

Countable set has measure 0

- Let $E = \{e_1, e_2, \ldots,\}$ be a countable set, $\epsilon > 0$.
- Then E can be covered with

$$F = \bigcup_{i=1}^{\infty} \left(e_i - \frac{\epsilon}{2^{i+1}}, e_i + \frac{\epsilon}{2^{i+1}} \right)$$

$$\text{total length of intervals in } F = \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$$

- Hence $\mathbb{Q}$ is measure 0.

$[0, 1]$ is not measure 0

- Suppose I_i is a collection of countably many intervals that cover $[0, 1]$.
- Because $[0, 1]$ is compact, I_i has a finite subcover: i.e. there exists a finite subcollection $(I_{n_k})_{k=1}^{L}$ such that

$$[0, 1] \subseteq \bigcup_{k=1}^{L} I_{n_k}.$$

- A finite collection of intervals that cover $[0, 1]$ must have total length greater than or equal to 1.

When is $\lim \int f_n dx = \int \lim f_n dx$?

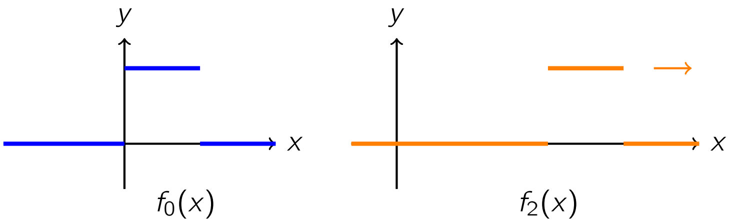
- Suppose $f_n(x)$ is the runaway function:

$$\text{If } f_n(x) := \mathbb{1}_{[n,n+1]}(x)$$

is the indicator function on $[n, n+1]$, then for all x,

$$f_n(x) \to 0 \text{ as } n \to \infty.$$

Note that this is pointwise convergence; we need larger n for larger x. When $n > x$, all of f_n would be 0 at x.



The Runaway Function

- For the runaway function,

$$\lim f_n = \mathbf{0}, \text{ but } \int f_n(x)dx = 1 \text{ for all } n.$$

Thus, when we take the integral of the left side and the limit of the right side, we get

$$\int \mathbf{0}(x)dx = 0 \neq 1 = \lim_{n\to\infty} 1$$

- We cannot exchange limit and integral!

The "Spike" Function

- Now suppose $f_n(x)$ is the spike function:

$$f_n(x) = n \cdot \mathbb{1}_{(0,\frac{1}{n})}(x)$$

is n times the indicator function on $(0, \frac{1}{n})$, then for all x,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \forall\, n,\ \text{if } x \leq 0 \\ 0 & \text{if } n > \frac{1}{x} \end{cases}$$

- However, the integral of f_n is 1 for all n.
- We still cannot exchange limit and integral:

$$\int \lim_{n\to\infty} f_n(x)dx = 0 \neq 1 = \lim_{n\to\infty} \int f_n(x)dx$$

The "Flattening" Function

- Again, suppose $f_n(x), n \in \mathbb{Z}^+$ is the flattening function:

$$f_n(x) = \frac{1}{n} \cdot \mathbb{1}_{(0,n)}(x)$$

  is $\frac{1}{n}$ times the indicator function on $(0, n)$, then for all x,

$$f_n(x) \leq \frac{1}{n} \, \forall \, x \implies \lim_{n \to \infty} f_n(x) = 0$$

- However, the integral of f_n is 1 for all n.
- Again we have

$$\int \lim_{n \to \infty} f_n(x) dx = 0 \neq 1 = \lim_{n \to \infty} \int f_n(x) dx$$

When is $\lim \int f_n dx = \int \lim f_n dx$?

- Our criterion for switching **lim** and $\int$ should exclude all of them.
- We want our function to converge pointwise. Here pointwise convergence can be relaxed to *almost everywhere* pointwise convergence, i.e. converge except on set of measure zero.
- **Caution:** Keep track of what integral you are using. You need Peano-Jordan measure 0 for Riemann integrals, and Lebesgue measure 0 for Lebesgue integrals - a sequence converging on $\mathbb{R} \setminus \mathbb{Q}$ but not on $\mathbb{Q}$ doesn't necessarily give you a Riemann integrable limit. The rationals $\mathbb{Q}$ has Lebesgue measure 0, but is not Peano-Jordan measurable.

When is $\lim \int f_n dx = \int \lim f_n dx$?

- The support of a function f is defined as

$$\operatorname{supp}(f) = \{x \mid f(x) \neq 0\}.$$

- Is it enough to say the support of the function is bounded? No - see the spike function.
- Is it enough to say the value of the function is bounded? No - see the runaway and the flattening function.
- What if we require both?

  That is sufficient, but not the best we can get : $\int_1^\infty \frac{1}{x^2} dx$ exists, and we in fact define it as the limit

$$\lim_{n\to\infty} \int_1^n \frac{1}{x^2} dx = \lim_{n\to\infty} \int \underbrace{\frac{1}{x^2} \cdot \mathbb{1}_{[1,n]}(x)}_{f_n(x)} dx$$

Monotone and Dominated Convergence, Riemann

Theorem (Monotone Convergence)

Suppose $\{f_n\}, f$ *are R-integrable functions such that*

$$0 \leq f_n(x) \leq f_{n+1}(x) < \infty \quad \forall x \in \mathbb{R}, n \in \mathbb{Z}^+.$$

Suppose $f(x) = \lim_n f_n(x)$ *is R-integrable. Then we have*

$$\int f(x)dx = \lim \int f_n(x)dx. \tag{1}$$

Theorem (Dominated Convergence)

Suppose $\{f_n\}, f$ *are R-integrable functions s.t.* $f_n \to f$ *pointwise,*

$$0 \leq |f_n(x)| \leq M \quad \forall x \in \mathbb{R}, n \in \mathbb{Z}^+,$$

and $\operatorname{supp}(f) \subset [a, b] s.t. -\infty < a < b < \infty$. *Then (1) also holds.*

Monotone and Dominated Convergence, Lebesgue

Theorem (Monotone Convergence)

Suppose $\{f_n\}$ are L-integrable functions such that

$$0 \leq f_n(x) \leq f_{n+1}(x) < \infty \quad \forall x \in \mathbb{R}, n \in \mathbb{Z}^+.$$

Suppose $f(x) = \lim_n f_n(x)$. Then we have

$$\int f(x)dx = \lim \int f_n(x)dx. \tag{2}$$

Theorem (Dominated Convergence)

Suppose $\{f_n\}$ are L-integrable functions s.t. $f_n \to f$ pointwise,

$$0 \leq |f_n(x)| \leq g(x) \quad \forall x \in \mathbb{R}, n \in \mathbb{Z}^+,$$

for a Lebesgue-integrable function $g(x)$. Then (2) also holds.

Monotone and Dominated Convergence

- For the Riemann integral version, you need to assume the limit function is R-integrable. For Lebesgue integrals, it's part of the conclusion the limit is L-integrable.
- What condition does the runaway function, the spike function, and the flattening function fail?

  None of them is monotone.

  For the Lebesgue version, the smallest envelope of runaway is $\mathbb{1}_{[0,\infty)}(x)$ and the smallest envelope of the spike and the flattening function is on the order of $\frac{1}{x}$. For the Riemann version, runaway and flattening has no bounded support while spike is not bounded.
- Why can't you R-integrate $\mathbb{1}_{\mathbb{Q}}$ as the pointwise limit of

$$\mathbb{1}_{\{r_1, r_2, \cdots, r_k\}}$$

  where $r_1, r_2, \cdots$ is an enumeration of $\mathbb{Q}$?

  Part of the assumption is the limit f is R-integrable.

Lebesgue integrals

- Provides a way of defining Lebesgue integrals while sweeping measure theory under the rug: A function f is Lebesgue integrable if there's R-integrable functions $\{f_n\}, \lim_{n\to\infty} f_n = f$, and you define the L-integral of f to be
$$\int f dx = \lim \int f_n dx.$$
- What's the Lebesgue integral of $\mathbb{1}_{\mathbb{Q}}$?

Theorem (Lebesgue's Criterion for Riemann Integrability)

A function f is Riemann integrable if it is only discontinuous on a set of Lebesgue measure 0.
Alternatively, if $\operatorname{osc}_f(I) := \sup_I(f) - \inf_I(f)$, then a function is Riemann integrable iff for any ϵ,

$$\exists \{I_i\}_{i=1}^n, \quad \bigsqcup_{i=1}^n I_i = \operatorname{supp}(f), \sum_{\operatorname{osc}_f(I_i)>\epsilon} \operatorname{vol}(I_i) < \epsilon.$$

L^1, L^2 functions

Let

$$L^1([0,1]) = \{f : [0,1] \to \mathbb{R} : \int_0^1 |f(x)| dx < \infty\}$$

$$L^2([0,1]) = \{f : [0,1] \to \mathbb{R} : \int_0^1 |f(x)|^2 dx < \infty\}$$

$L^1([0,1])$ and $L^2([0,1])$ have norms, namely $\|f\|_1 = \int_0^1 |f(x)| dx$ and

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2}.$$

Notice the similarity to $(\mathbb{R}^n, |\cdot|_1)$ and $(\mathbb{R}^n, |\cdot|_2)$ where

$$|x|_1 = |x_1| + |x_2| + \cdots + |x_n|, |x|_2 = \sqrt{x_1^2 + \cdots + x_n^2}.$$

L^1 and L^2 functions

Norm $\|f\|_1$ and $\|f\|_2$ does satisfy the triangle inequality:

$$\int_0^1 |f(x)+g(x)|dx \leq \int_0^1 |f(x)|dx + \int_0^1 |g(x)|dx$$

$$\begin{aligned}\left(\int_0^1 |f(x)+g(x)|^2 dx\right)^{1/2} &\leq \left(\left(\int_0^1 |f(x)+g(x)|dx\right)^2\right)^{1/2} \\ &\leq \left(\int_0^1 |f(x)|^2 dx\right)^{1/2} + \left(\int_0^1 |g(x)|^2 dx\right)^{1/2}\end{aligned}$$

where the first step of the L^2 equality follows from the Cauchy-Schwarz inequality: $\|fg\|_1 \leq \|f\|_2 \|g\|_2$.

L^1 and L^2 spaces

- A norm must be:
  - Subadditive: $N(u+v) \leq N(u) + N(v)$ (Triangle ineq.)
  - Absolutely scalable: $|\alpha| N(u) = N(\alpha u)$, α is a scalar.
  - Positive definite/point-separating/nondegeneracy:
    $$N(u) = 0 \text{ iff } u = \mathbf{0}.$$
- But $\|\cdot\|_1$ and $\|\cdot\|_2$ don't quite satisfy "nondegeneracy."

  For if $f(x) = g(x)$ almost everywhere, then

  $$\|f\|_1 = \|g\|_1 \text{ and } \|f\|_2 = \|g\|_2.$$

  So we'll define an equivalence class that declares two functions equal if they are equal almost everywhere.

Facts about L^1

- The limit of L^1 functions is L^1. That is, if (f_k) are a sequence of L^1 functions whose norm $\|f_m - f_n\|_1$ gets arbitrarily small for all m and n large enough, then there exists an L^1 function f such that $\|f_n - f\|_1 \to 0$ as $n \to \infty$.
- So L^1 is "complete."
- True for L^2 as well.
- L^2 is equipped with an "inner product"

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx,$$

with $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$, $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$,

$$|\langle f, g\rangle| = |\langle f, f\rangle|^{1/2}|\langle g, g\rangle|^{1/2} < \infty.$$

Inner Product Space

An inner product space is a vector space V over a field $F = \mathbb{R}$ or $\mathbb{C}$ with an inner product $\langle \cdot, \cdot \rangle : V \times V \to F$:

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (the complex conjugate).
- $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ where $a, b \in F$.
- $\langle x, x \rangle > 0$ if $x \in V \setminus \{\mathbf{0}\}$.

A complete inner product space is a Hilbert space.

- If you have an orthonormal basis B in a Hilbert space, which means

  $$\langle b, b \rangle = 1, \langle b, b' \rangle = 0 \forall b \neq b' \in B$$

  then we can represent everything by the basis:

  $$x = \sum_{b \in B} \langle x, b \rangle b, \langle x, x \rangle = \sum_{b \in B} \langle x, b \rangle^2 \tag{3}$$

A Fun Interplay of L^1 and L^2 Spaces

- The Fourier transformation for $f \in L^2([0,1])$ is defined using the L^2 inner product:

$$\hat{f}(\xi) = \langle f, e^{2\pi i \xi x} \rangle = \int_{\mathbb{T}} f(x) e^{2\pi i \xi x} dx, \mathbb{T} = [0,1]/_{0\sim 1}$$

- The property (3) becomes Plancherel's theorem:

$$\int_{\mathbb{T}} |f(x)|^2 dx = \int_{\mathbb{T}} |\hat{f}(\xi)|^2 d\xi$$

Or, discretely, when $\lim_n \|f(x) - s_n(x)\|_2 \to 0$, where $s_n(x)$ is $\sum_{|i|\leq n} \hat{f}(n) e^{2\pi i n x}$ we have Parseval's identity:

$$\int_{\mathbb{T}} |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$$

A Fun Interplay of L^1 and L^2 Spaces

- For square integrable functions, i.e. L^2 functions, we can naturally define $\langle f, e^{2\pi i\xi x}\rangle$ and partial sums of the Fourier series of f converges to something (not necessarily f) in L^2 because they form a Cauchy sequence there.
- However, the integral defining the Fourier coefficients $\hat{f}(\xi)$ can only be evaluated when it is absolutely integrable, i.e.

$$\int_{\mathbb{T}} |f(x)e^{2\pi i\xi x}|dx < \infty.$$

Notice that

$$|f(x)e^{2\pi i\xi x}| = |f(x)||e^{2\pi i\xi x}| = |f(x)|.$$

- This is essentially saying f needs to be L^1!
But $L^1(\mathbb{T}) \neq L^2(\mathbb{T})$.