Equidistribution Review 2

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July 5, 2020

Trig Weierstrass

- Recall Weierstrass Approximation:

Theorem

Every continuous function on an interval can be uniformly approximated by polynomials. That is, for each $f \in C([a, b])$ *and* $\epsilon > 0$ *there exists a polynomial* P *such that for all* $x \in [a, b]$*,*

$$|f(x) - P(x)| < \epsilon$$

- We will give a sketch of Trig Weierstrass Approximation:

Theorem

For each $f \in C([a, b])$ *and* $\epsilon > 0$ *there exists a trigonometric polynomial* P *such that for all* $x \in [a, b]$*,*

$$|f(x) - P(x)| < \epsilon$$

Remarks

- WLOG can assume $[a, b] = [-1, 1]$
- Can extend f to be a continuous function $[-\pi, \pi]$ such that $f(\pi) = f(-\pi)$.
- A trigonometric polynomial is a function of the form

$$\sum_{n=0}^{N} a_n \cos(nx) + b_n \sin(nx)$$

or if you prefer

$$\sum_{n=-N}^{N} a_n e^{ix}$$

Recall Proof of Weierstrass Approximation

- Suppose $[a, b] = [-1, 1]$. Recall that we proved the Weierstrass Approximation Theorem by taking a suitable sequence of polynomials K_n such that $K_n(0) \to \infty$, $K_n(x) \to 0$ for all other x, $K_n(x) \geq 0$ for all $x \in [-1, 1]$, and

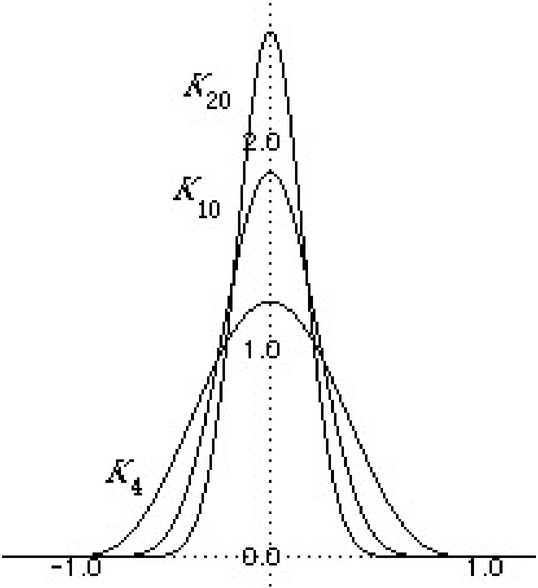
$$\int_{-1}^{1} K_n(x) = 2.$$

- We then convolved those functions by f to obtain a sequence of polynomials:

$$P_n(x) = f * K_n(x) = \int_{-1}^{1} f(t)K(x-t)dt$$

- The sequence of polynomials P_n converges uniformly to f.

Picture of K_n



http://www-groups.mcs.st-andrews.ac.uk/˜john/analysis/Lectures/L19.html

Sketch of Trig Weierstrass Approximation

- We follow a similar strategy: we find a sequence of β_n with
  - $\beta_n(0) \to \infty$
  - $\beta_n(x) \to 0$ for $x \neq 0$
  - $\beta_n(x) \geq 0$
  - $$\int_{-\pi}^{\pi} \beta_n(x) = 2\pi$$
  - $\beta_n(x)$ are trigonometric polynomials: or at least, when convolved with a continuous function f, they become a trigonometric polynomial:
$$T_n(x) = f * \beta_n(x) = \int_{-\pi}^{\pi} f(t)\beta_n(x-t)dt$$
- One β_n that works is
$$\beta_n(x) = \frac{1}{n}\frac{\sin^2(nx/2)}{\sin^2(x/2)}$$

[Time permitting] A brief aside

- The function f has a Fourier series:

$$\sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

Taking

$$S_N f(x) = \sum_{n=0}^{N} a_n \cos(nx) + b_n \sin(nx) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

and $A_N f(x) = \frac{1}{N} \sum_{i=0}^{N-1} S_N f(x)$, the quantity $T_n(x) = A_n(x)$.

[Time permitting] Partial Fourier Series Doesn't Work

- In fact, we have nth Fourier trig polynomial

$$S_N f(x) = f * \gamma_n(x), \quad \gamma_n(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)}$$

- It turns out that while $\int_{-\pi}^{\pi} \gamma_n(x) dx = 2\pi$,

$$x_n = \int_{-\pi}^{\pi} |\gamma_n(x)| dx \text{ is unbounded.}$$

  while since $\beta_n(x) \geq 0$,

$$\int_{-\pi}^{\pi} |\beta_n(x)| dx = \int_{-\pi}^{\pi} \beta_n(x) dx = 2\pi.$$

- $S_N f(x)$ only converges "pointwise almost everywhere" (not even pointwise) to $f(x)$ as $N \to \infty$. It does not converge uniformly to $f(x)$.

Metric Spaces

- A metric space is a set X equipped with a *metric* ("distance function") d satisfying the following properties:
  - Positivity: $d(x, y) \geq 0$
  - Nondegeneracy: $d(x, y) = 0 \implies x = y$ for $x, y \in X$
  - Reflexivity: $d(x, y) = d(y, x)$
  - The triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$

Examples of Metric Spaces

- Distance between two real numbers: $X = \mathbb{R}$, $d(x, y) = |x - y|$
- Euclidean distance: $X = \mathbb{R}^2$

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

- Euclidean distance for $\mathbb{R}^n$: $X = \mathbb{R}^n$

$$d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

where $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{y} = (y_1, y_2, \ldots, y_n)$.

Proof of Euclidean Metric

- Positivity: Principle square root is positive.
- Nondegeneracy: $d(x, y) = 0$ means

  $$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = 0$$

  since $(x_i - y_i)^2 \geq 0$ and is 0 if and only if $x_i = y_i$, it follows that $x_i = y_i$ for all i.
- Reflexivity: Follows since $(x_i - y_i)^2 = (-(y_i - x_i))^2 = (-1)^2(y_i - x_i)^2 = (y_i - x_i)^2$
- Triangle Inequality: We will show $\|x\|_2 := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ satisfies

  $$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2.$$

Proof Continued

$$\sqrt{(x_1+y_1)^2+\cdots+(x_n+y_n)^2} = \sqrt{x_1^2+\cdots+x_n^2}+\sqrt{y_1^2+\cdots+y_n^2}$$

Squaring both sides, we have

$$(x_1+y_1)^2+\cdots+(x_n+y_n)^2 \leq x_1^2+\cdots+x_n^2+y_1^2+\cdots y_n^2+2\|x\|_2\|y\|_2$$

Expanding both sides and cancelling out like terms, we get

$$\sum_{i=1}^{n} x_i y_i \leq \|x\|_2\|y\|_2 = \sqrt{x_1^2+\cdots+x_n^2}\sqrt{y_1^2+\cdots+y_n^2}.$$

This is the "Cauchy-Schwarz inequality." Replacing x with $y-x=(y_1-x_1,\ldots,y_n-x_n)$, y with $z-y$, we see that $y-x+z-y=z-x$, and since $\|y-x\|=d(x,y)$, the triangle inequality follows.

Normed Metric Spaces

- A Norm is a way of measuring size in a "vector space" (think $\mathbb{R}^n$)
- A norm $\|\cdot\|$ satisfies the following properties:
  - Positivity: $\|x\| \geq 0$
  - Nondegeneracy: $\|x\| = 0$ if and only if $x = 0$
  - Scaling: $\|\lambda x\| = |\lambda|\|x\|$ for $\lambda \in \mathbb{R}$
  - Triangle Inequality: $\|x + y\| \leq \|x\| + \|y\|$.
- Given a norm, can form a distance $d(x, y) = \|x - y\|$.
- $\mathbb{R}^n$ under the "ℓ^2" or Euclidean norm $\|x\|_2$ is a Normed space.

L^p spaces

- L^p distance for $\mathbb{R}^n$ for $1 \leq p \leq \infty$: $X = \mathbb{R}^n$
- $$\|\vec{x}\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + \cdots + |x_n|^p}.$$
- $$\|\vec{x}\|_1 = |x_1| + \cdots + |x_n|$$
- $$\|\vec{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$
- To prove these, use "Hölder's inequality" in place of Cauchy-Schwarz inequality. (look at the counting measure section on the wikipedia page)

Other Examples of Metric Spaces

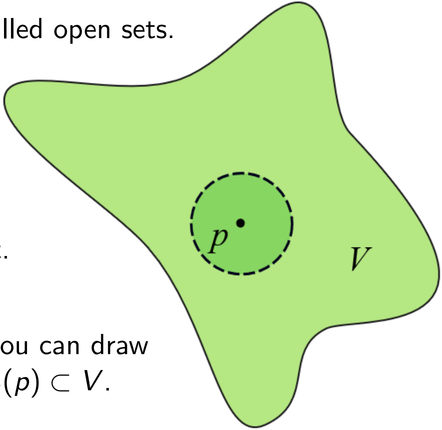
- A subspace of a metric space.
- Trivial metric on any set: $d(x, y) = 1$ for all $x, y \in S$
- A multiple of a metric space: if d is a metric, then $2d$ is also a metric.
- Let $X = \{0, 1\}^{\mathbb{N}}$ the space of all infinite sequences $x = (x_1, x_2, x_3, x_4, \dots)$ with x_i is either 0 or 1. Define

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i} = \frac{|x_1 - y_1|}{2} + \frac{|x_2 - y_2|}{2^2} + \cdots$$

- More on this example later.

Topology - Open sets

Certain subsets of a space X are called open sets.



Intuition

In a open set, no one is at the border - you can always draw a little disk around yourself such that the disk is completely inside the set.

Definition

A set V is a neighborhood of p if you can draw a disk $B_r(p)$ around p such that $B_r(p) \subset V$.

An open set is a neighborhood of all of its points.

Image Credit: Oleg Alexandrov / Public domain.

https://commons.wikimedia.org/wiki/File:Neighborhood_illust1.svg

Metric Space Terminology

- Open balls: $B_r(x_0) = \{x : d(x, x_0) < r\}$.
- Closed balls: $\overline{B}_r(x_0) = \{x : d(x, x_0) \leq r\}$.
- Open sets U. A subset U of X is open if for all $x \in U$, there exists r such that $B_r(x) \subset U$.
- $(0, 1)$ is open in $\mathbb{R}$. While $[0, 1]$ is not open in $\mathbb{R}$ (look at the endpoints).
- Closed sets contain all its <u>limit points</u>. $[0, 1]$ is closed. $[0, 1)$ is neither closed nor open.
- Another definition of Closed Sets: complements of open sets

More examples of Open and Closed Sets

- Any union of open intervals are open in $\mathbb{R}$
- The entire space and the empty set are open and closed for any metric space.
- The trivial metric on any set: $d(x, y) = 1$ for all $x, y \in S$: all subsets of S are open and closed.
- Finite union of closed intervals are closed.
- $\{0\}$ is not an open subset of $\mathbb{R}$.

Properties of Open and Closed Sets

- For a metric space (M, d), M and $\emptyset$ are both open and closed.
- The union of any number (countable, uncountable) of open sets is open.
- The intersection of any number (countable, uncountable) of closed sets is closed.
- The finite union of closed sets is closed.
- The finite intersection of open sets is open.
- Infinite intersections of open sets are not open:

$$\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$$

which is not open.

Continuity: $\epsilon - \delta$ Definition

- A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0 if for all $\epsilon > 0$, there exists δ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

Letting $d(x, y) = |x - y|$, we can reformulate this:

$$\boxed{d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \epsilon.}$$

- A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if it's continuous at all points.
- A function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if given $\epsilon > 0$ there exists δ such that for all $x, y \in \mathbb{R}$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

- Can reformulate for metric:

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon.$$

Continuity

- Given two metric spaces (X, d_X), (Y, d_Y), a function $f : X \to Y$ is continuous at a point x_0 if for all $\epsilon > 0$ there exists δ such that

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon. \qquad (*)$$

- A function $f : X \to Y$ is continuous if it's continuous at all points.
- Similarly, a function $f : X \to Y$ is uniformly continuous if given any $\epsilon > 0$, there exists δ such that for all $x, y \in X$, $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$.

Continuity

Equivalently,

Definition

1. A function between two metric spaces is continuous if every preimage of an open set is open.
2. A function between two metric spaces is continuous if every preimage of an open ball is open.

- $1 \implies 2$: all open balls are open sets.
- $2 \implies 1$: every open set is a (arbitrary) union of open balls, and the preimage of a union is the union of preimages.
- $U = B_\epsilon(f(x_0))$ is an open ball, and the definition (*) says
$$f\left(B_\delta(x_0)\right) \subset U.$$

Limit Points

Definition

A point p is a limit point of a set S if every open set containing p also contains a point $q \in S$ such that $q \neq p$.

- 1 is a limit point of $[0, 1)$.
- 1 is not a limit point of $\{1, 2, 3, \ldots\}$. Take $B_{1/2}(1) = \left(\frac{1}{2}, \frac{3}{2}\right)$.
- Every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$. In particular every $x \in \mathbb{Q}$ is a limit point of $\mathbb{Q}$.

Recall Closed sets contain all of its limit points.

- If we take a set S and union it with the set of limit points of S, we get its closure, denoted $\overline{S}$.
  Nontrivial fact: The set $\overline{S}$ contain all of its limit points.

Definition

A point p is an isolated point of a set S if $p \in S$ but is not a limit point of S.

Cauchy Sequences

Definition

Take a metric space (M, d). A sequence $\{a_i\}_{i=1}^{\infty}$ in it is Cauchy if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } \forall m, n > N, |a_m - a_n| < \epsilon. \tag{1}$$

Equivalently:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, x \in M, \text{ s.t. } \forall m > N, a_m \in B_\epsilon(x). \tag{2}$$

Proof of equivalence. Fix $\epsilon > 0$.

- (1) $\implies$ (2). Take the N for ϵ from (1). Let $x = a_{N+1}$.

$$\forall m \in \mathbb{N}, \quad a_m \in B_\epsilon(a_{N+1}).$$

- (2) $\implies$ (1). Take the N and x for $\frac{\epsilon}{2}$ from (2). $\forall m, n \in \mathbb{N}$,

$$a_m, a_n \in B_{\epsilon/2}(x) \implies |a_m - a_n| < \text{diameter}(B_{\epsilon/2}(x)) = \epsilon.$$

Complete Metric Space

- Cauchy sequences provides a way to characterize "converging" sequences without having the construct the limit first.

Definition
A sequence $\{a_i\}_{i \in \mathbb{N}}$ converges to x if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } \forall m > N, d(a_m, x) < \epsilon.$$

Definition
A metric space (M, d) is complete if every Cauchy sequence in it converges to a point in it.

Close vs Complete

Closure of subsets is relative to the entire set, while completeness of a metric space is relative to the metric.

Take the Euclidean distance metric:

- Every set is closed relative to itself. The open interval $(0, 1)$ is closed relative to $(0, 1)$, but not relative to $[0, 1)$.
- $\mathbb{Q}$ is closed relative to $\mathbb{Q}$.
- $\mathbb{Q}$ is not closed relative to $\mathbb{R}$.
- $\mathbb{Q}$ is not complete, $\mathbb{R}$ is complete, $\mathbb{R}^n$ is complete for $n \in \mathbb{N}$.
- $\mathbb{R}$ is closed relative to $\mathbb{R}^n$ for all $n \in \mathbb{N}$.

Fact. A metric space is complete iff it is closed in every space containing it.

Compact Metric Spaces

Definition

A metric space is compact if for all sequences (x_n), there exists a subsequence of (x_n) that converges to an element of that metric space.

- $[0, 1]$ is compact (Try to find a monotone subsequence for any sequence!)
- $(0, 1)$ is not compact: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a subsequence that does not converge.
- All closed and bounded subsets of $\mathbb{R}^n$ are compact. All other sets are not compact.
- $\mathbb{R}^{\mathbb{N}}$ is not compact. Let $\vec{e_i}$ be the i^{th} unit vec; take $\{\vec{e_i}\}_{i\in\mathbb{N}}$.
- It turns out that the space $\{0, 1\}^{\mathbb{N}}$ is compact.(What metric?)

Continuity Revisited

Theorem
Every continuous function on a compact set is uniformly continuous.

Theorem
Every continuous function on a compact set achieves its supremum and infimum (thus we speak of their *maximum* and *minimum* in this case.

Theorem
Every continuous function maps compact sets to compact sets.

Cantor's Intersection Theorem

Theorem

Take a space S. The intersection of nested compact, closed subsets of S is nonempty.

Example

- Q has no compact subsets.
- Compare
$$\bigcap_{n\in\mathbb{N}} \left[0, \frac{1}{n}\right] \text{ vs } \bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right)$$
- Consider $\{S_k\}_{k\in\mathbb{N}}, S_k \subset \mathbb{R}^{\mathbb{N}}$ where
$$S_k := \{0\}^k \times [0,1]^{\mathbb{N}}$$
- Take $\mathbb{R}$. The intersection of nested closed, bounded subsets of $\mathbb{R}$ is nonempty. Consider the construction of Cantor set.

Cantor Set

The Cantor set can be defined by:

$$C_0 := [0,1], \quad C_n := \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3} \right) \forall n \in \mathbb{Z}^+$$

$$\mathcal{C} := \bigcap_{i=0}^{\infty} C_i$$

- C_n are all closed - a union of 2^n closed intervals.
- WTS: C_n are nested, i.e. $C_n \subset C_{n-1} \forall n \in \mathbb{Z}^+$.

True for $n = 1$; By WOP, the least bad $n \geq 2$ and $C_{n-1} \subset C_{n-2}$.

$$\frac{C_{n-1}}{3} \subset \frac{C_{n-2}}{3}, \quad \left(\frac{2}{3} + \frac{C_{n-1}}{3} \right) \subset \left(\frac{2}{3} + \frac{C_{n-2}}{3} \right)$$

Cantor Set

- Elements of $\mathcal{C}$ are exactly those with only digit 0 or 2 in (one of) their ternary expansion.

$$\mathcal{C} = \left\{ \sum_{i=1}^{\infty} \frac{t_i}{3^i} \;\middle|\; t_i \in \{0, 2\} \right\}.$$

- We can also see the elements of Cantor set as infinite $0, 2-$sequences.
- Cantor set is closed and bounded in $\mathbb{R}$, thus compact.
- Cantor set is a complete metric space w.r.t. the absolute value metric.
- Cantor set has no isolated points.
- What is

$$\mathcal{C} + \mathcal{C} := \{a + b \mid a, b \in \mathcal{C}\}?$$

Cantor Set vs $\{0,1\}^{\mathbb{N}}$

Definition

Two spaces X, Y are homeomorphic if we have a continuous map $X \to Y$ with a continuous inverse.

Theorem

The Cantor set is homeomorphic to $\{0,1\}^{\mathbb{N}}$ if we give $\{0,1\}^{\mathbb{N}}$ the metric

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

- The natural map we'll use:

$$f : \mathcal{C} \to \{0,1\}^{\mathbb{N}}, \quad (t_1, t_2, t_3 \ldots) \mapsto \left(\frac{t_1}{2}, \frac{t_2}{2}, \frac{t_3}{2} \ldots \right)$$

Cantor Set vs $\{0,1\}^{\mathbb{N}}$

Proof.

- ***f* is continuous:** Take $\epsilon > 0$ and $f(x) \in \{0,1\}^{\mathbb{N}}$. If $\frac{1}{2^k} < \epsilon$, then everything who agrees with x up to digit k is within ϵ from x.

  Take $\delta = \frac{1}{3^k}$ ensures the preimage agrees with x up to k digits.

- **f^{-1} is continuous:** Similarly, take k such that $\epsilon > \frac{1}{3^k}$. The $\delta = \frac{1}{2^k}$ would suffice.

$\square$

Definitions of U.D.

1. A sequence $(x_n) \in [0,1]$ is *uniformly distributed* (U.D.) if for any subinterval $[a,b]$ one has

$$\lim_{N\to\infty} \frac{|\{x_n : 1 \le n \le N\} \cap [a,b]|}{N} \to b-a.$$

Definition

2. A sequence $(x_n) \in [0,1]$ is uniformly distributed if for all $f \in C^0([0,1])$, i.e. continuous functions $f : [0,1] \to \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i) = \int_0^1 f(x)dx. \tag{3}$$

Definition

3. A sequence $(x_n) \in [0,1]$ is uniformly distributed if for all Riemann integrable f, (3) holds.

Equivalence of definitions

- 2. $\implies$ 1. All indicator functions are continuous. Note that

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{[a,b]}(x_i) = \frac{|\{x_n : 1 \le n \le N\} \cap [a,b]|}{N}$$

- 3. $\implies$ 2. All continuous functions are Riemann integrable.
- 1. $\implies$ 3. Riemann integrable functions can be approximated by Jordan-Simple functions, and Jordan-Simple functions can be approximated by step functions. Allow $\epsilon/3$ error for each of the above two approximations
For the step functions $\sum_{i=1}^{n} c_i \mathbf{1}_{[s_i,t_i]}(x)$, take N_i for each i s.t.

$$\left| \frac{|\{x_n : 1 \le n \le N\} \cap [s_i,t_i]|}{N} - (t_i - s_i) \right| \le \frac{\epsilon}{3 \cdot 2^i c_i}, \quad \forall N > N_i$$

and take $\tilde{N} = \max_i N_i$.

The set of terms vs U.D. of the sequence

- You can enumerate $\mathbb{Q} \cap (0, 1]$ so that it is U.D.

  Write rationals as $\frac{a}{b}$ for coprime a, b and arrange them by lexicographical order on the pair (b, a). Equivalently, arrange by the order they are inserted in a Farey sequence.

- You can enumerate $\mathbb{Q} \cap (0, 1]$ so that it is not U.D.

  Take the sequence from above, break it into two subsequences with terms in $(0, \frac{1}{2}]$ and in $(\frac{1}{2}, 1]$, then merge the two subsequences by taking two terms in the first sequence for every term in the second sequence.