Ross Measure Theory Talk

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Summary

- Part 1: Riemann integrability-will discuss Riemann integration
- Part 2: Lebesgue Measures and Lebesgue Integrals-Will discuss Lebesgue measure

Part 1: Riemann integrability

Measure Theory

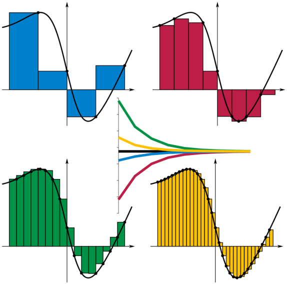
- Measure theory is the study of area.
- Rigorously define area and extend the "area under the curve" notion of integration.

Some notation

- inf is "greatest lower bound" and sup is "least upper bound" (e.g. $\inf(0,1) = 0$, $\sup(0,1) = 1$).
- Unlike integers, bounded subsets of the real numbers don't have a min or a max, so we use inf and sup, respectively, instead.
- $\int_E f(x)dx$ indicates integration along E, or integration as x ranges in E.
- $$1_E(x) := \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

Finding the Area under the graph

- Let $f : [a, b] \to \mathbb{R}$ be a function. Suppose want to find "area under the curve of f."
- Approximate the area under f with smaller and smaller rectangles whose base is on the x-axis and height is f evaluated at an element of the base.
- And you take the limit as you decrease the width of those rectangles.



(Taken from Wikipedia)

Riemann Sums

- $P = \{x_1, x_2, \ldots, x_n : a = x_1 < x_2 < \cdots < x_n = b\}$ a "partition."

$$L(P,f) = \sum_{i=1}^{n-1} \min\{f(x) | x_i \le x < x_{i+1}\}(x_{i+1} - x_i)$$

$$U(P,f) = \sum_{i=1}^{n-1} \max\{f(x) | x_i \le x < x_{i+1}\}(x_{i+1} - x_i)$$

- $L(f) = \sup_P L(P,f)$, $U(f) = \inf_P U(P,f)$ take sup and inf over all partitions.
- f is integrable if $L(f) = U(f)$.
- Stewart's definition: practical use (only use for continuous functions):

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(a + (b-a)i/n)$$

Continuous Function

Speaking of Which, a function $f : [a, b] \to \mathbb{R}$ is continuous if

$$\lim_{x \to c} f(x) = f(c).$$

The set of all continuous functions over $\mathbb{R}$ is denoted $C([a, b])$.

Example

Take $f(x) = x$. Then limit above is

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n} = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \frac{1}{2}.$$

Failure of Riemann Integrability

- Integral = "area under the curve." Draw the graph of a function and just look at the area it bounds.
- Makes sense for continuous functions but doesn't make sense for functions like

$$1_{\mathbb{Q}} := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

- We'll introduce a different type of integral, the Lebesgue integral, that expands on the Riemann integral that allows us to integrate such functions.

Riemann-Lebesgue Theorem

- A set E is a zero set if for each $\epsilon > 0$ there exists intervals $(I_i = [a_i, b_i])_{i=1}^{\infty}$ with

$$E \subseteq \bigcup_{i=1}^{\infty} I_i \quad \sum_{i=1}^{\infty}(b_i - a_i) < \epsilon$$

Theorem

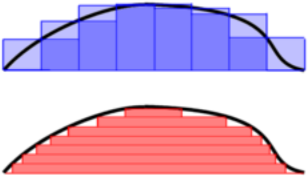
$f : [a, b] \to \mathbb{R}$ is integrable if and only if the set of discontinuities of f is a zero set.

Turning the rectangles

- Suppose $c \leq f(x) \leq d$
- Q a partition of range, $c = y_1 < y_2 < \cdots < y_n = d$
- Let $A_i = \{x \in [a, b] : y_i \leq f(x) \leq y_{i+1}\}$
- "Side Riemann sums"

$$S(Q) = \sum_{i=1}^{k-1} y_i \mu(A_i)$$

μ will be defined later.



(Taken from Wikipedia)

Jordan Measures

- Riemann integrability corresponds to *Jordan Measures*.
- We'll define the Jordan measure to be a function μ from "Jordan-measureable sets" to $[0, \infty]$ with

$$\mu([a, b)) = b - a$$

- For an arbitrary set S, the Jordan outer-measure is

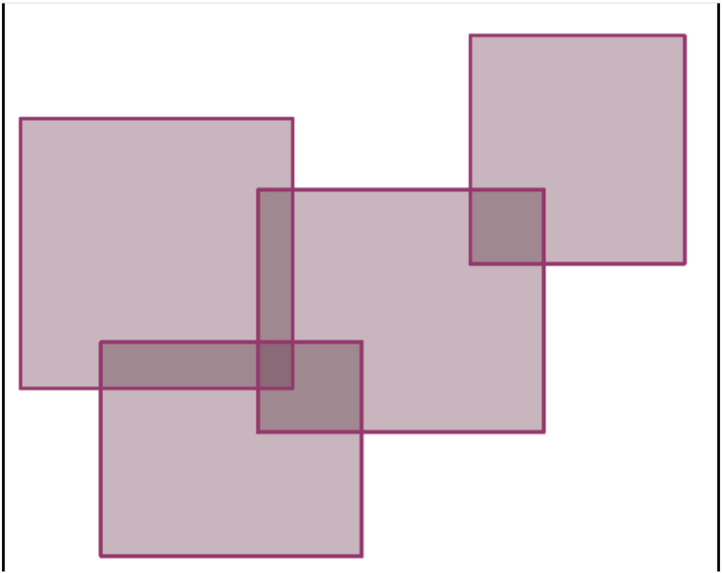
$$\mu^*(S) = \inf_{I_i} \sum_{i=1}^{n} \mu(I_i)$$

  for I_i finite set of intervals that cover S and the inner measure is

$$\mu_*(S) = \sup_{I_i} \sum_{i=1}^{n} \mu(I_i)$$

  I_i finite set of intervals that is contained in S.
- A set is Jordan-measurable if and only if $\mu^*(S) = \mu_*(S)$ and we define the Jordan measure $\mu(S) = \mu^*(S)$.
- $\mathbb{Q}$ is not Jordan measurable.

Jordan Measure



Properties of the Jordan Measure

- $\mu(\emptyset) = 0$
- $\mu(A \cup B) \leq \mu(A) + \mu(B)$
- $A \subset B \implies \mu(A) \leq \mu(B)$
- $\mu(A) \geq 0$

Riemann Integrals

- A Jordan-*Simple function* is a sum

$$h(x) = \sum_{i=1}^{n} a_i 1_{A_i}(x)$$

where 1_{A_i} is the indicator function for the set A_i and A_i are Jordan measurable.

- We define the Riemann integral of h as

$$\int h(x)dx = \sum_{i=1}^{n} a_i \mu(A_i).$$

- A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if there exists a sequence of Jordan simple functions f_i that approximate it and whose integrals f_i converge.
- Compare to the usual definition of using left and right sums.

Part 2: Lebesgue Measure and Lebesgue Integral

Lebesgue Measure

- We define the Lebesgue outer measure as

$$\mu^*(S) = \inf\{\sum_{i=1}^{\infty} \mu(I_i) : \bigcup I_i \supseteq S\}$$

  and inner measure similarly.
- Difference between Lebesgue and Jordan is you allow infinitely many intervals whereas Jordan only allows finitely many.
- Define Lebesgue measurable sets similarly: where outer and inner measures coincide.
- Approximated by Lebesgue simple functions $\implies$ Can take its integral.

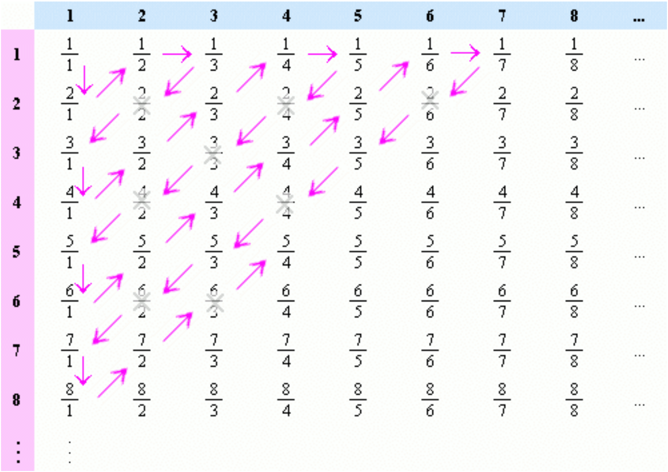
Properties of Lebesgue Measure

- $\mu(\emptyset) = 0$
- $A \subset B \implies \mu(A) \leq \mu(B)$
- $\sum \mu(A_i) \geq \mu(\bigcup A_i)$ "countable subadditivity" (if A_i are disjoint then inequality is equality)
- The last point is the only difference between Lebesgue and Jordan: that you can take "countable" sums.
- A function satisfying these general axiom for sets in a measureable space is known as a measure.

Countable?

- Look up numberphile video “infinity is bigger than you think.”
- Means you can list them out.
- $\mathbb{N}$ is countable: $1, 2, 3, 4 \ldots$
- $\mathbb{Z}$ is countable: $0, 1, -1, 2, -2, 3, -3, \ldots$
- $\mathbb{R}$ is not countable (Cantor diagonal argument)

$\mathbb{Q}$ is countable



(from `https://math.stackexchange.com/questions/501782/is-the-infinite-table-argument-for-the-countability-of-q-u`

Measure of a Countable set is 0

- Let $E = \{e_1, e_2, \dots,\}$ be a countable set, $\epsilon > 0$.
- Then E can be covered with

$$F = \bigcup_{i=1}^{\infty} \left[e_i - \frac{\epsilon}{2^{i+1}}, e_i + \frac{\epsilon}{2^{i+1}}\right]$$

  and by countable subadditivity,

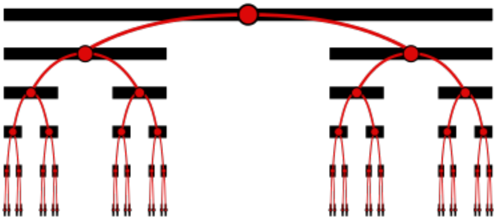
$$\mu(F) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$$

  and since $E \subset F$, $\mu(E) \leq \mu(F) \leq \epsilon$.
- Hence $\mu(\mathbb{Q}) = 0$.

$$\int 1_{\mathbb{Q}}(x)dx = 0.$$

Cantor Set

- Take $[0, 1]$.
- Take away middle third $(1/3, 2/3)$. End up with $[0, 1/3] \cup [2/3, 1]$.
- Repeat for each of those two intervals.



(Taken from Wikipedia)

- Let C be the Cantor set. It is “totally disconnected” (it has no intervals of positive length in it) and has uncountably many elements: it consists of all numbers between 0 and 1 whose base 3 expansion only has 0’s and 2’s.

Measure of Cantor Set

- $\mu([0,1]) = 1$
- $\mu([0,1] - (1/3, 2/3)) = 1 - 1/3$
- $\mu([0,1] - (1/3, 2/3) - (1/9, 2/9) - (7/9, 8/9)) = 1 - 1/3 - 2/9$

$$\mu(C) = 1 - \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3} \right)^n = 0.$$

Fat Cantor Set

- Take $[0, 1]$
- Remove middle $1/4$ to get $[0, 3/8] \cup [5/8, 1]$
- Remove middle $1/4^n$ from each 2^{n-1} remaining intervals.

  (Taken from Wikipedia)
- "Looks like" (or is homeomorphic to, the technical term) the Cantor set.

Measure of Fat Cantor Set

$$\mu(C_{fat}) = 1 - \sum_{i=1}^{\infty} \frac{2^{i-1}}{4^i} = \frac{1}{2} > 0$$

Further Reading

- *Real Mathematical Analysis* by Pugh
- *Real and Complex Analysis* by Rudin
- *Real and Functional Analysis* by Lang
- *Measure and Category* by John Oxtoby