The Mapping Class Group acts continuously on the Moduli space of Marked Hyperbolic Structures of the First Kind

Young Geometric Group Theory X

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Definition (Teichmüller space)

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where $(X_1, f_1) \sim (X_2, f_2)$ iff there is an orientation preserving isometry $I : X_1 \to X_2$ isotopic to $f_2 \circ f_1^{-1}$.

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- The full mapping class group acts on it, rather than only the subgroup of mapping classes realisable by quasiconformal homeomorphisms of the 'basepoint' Riemann surface.
- Theorem (Thurston): There is an earthquake map between any two marked hyperbolic structures of the first kind.

Topology of the marked moduli space

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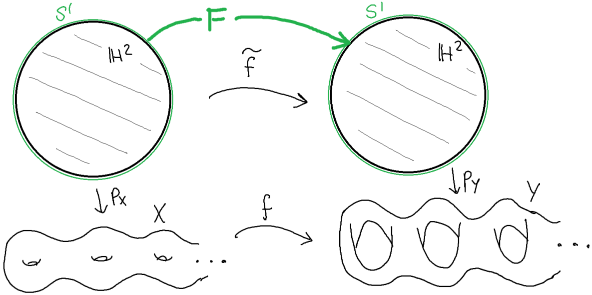
Lemma (Homeomorphism at infinity)

Let X, Y be complete hyperbolic surfaces of the first kind. Let $f : X \to Y$ be a homeomorphism with lift $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$ to universal covers. Then $\tilde{f}$ extends to a homeomorphism $\partial\tilde{f} : S^1 \to S^1$ at infinity, which is deck group equivariant. Further, if f_t is an isotopy which lifts to isotopy $\tilde{f}_t$, then $\partial\tilde{f}_0 = \partial\tilde{f}_1$.



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$$\begin{array}{ccc} S^1 & \xrightarrow{F} & S^1 \\ & & \\ \mathbb{H}^2 & \xrightarrow{\tilde{f}} & \mathbb{H}^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_X} \\ S & \xrightarrow{f} & X \end{array}$$

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The topology of $\mathcal{T}(S)$ is the pullback of the topology on the right. That is, declare Φ_p to be a homeomorphism.

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Next: What is the geometry of $\mathcal{T}(S)$?