A MODULI SPACE OF MARKED HYPERBOLIC STRUCTURES FOR BIG SURFACES

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ABSTRACT. We introduce the moduli space of marked, complete, Nielsen-convex hyperbolic structures on a surface of negative, but not necessarily finite, Euler characteristic. The emphasis is on the case in which the surface is of infinite type, the aim being to study the mapping class group of such a surface via its action on this marked moduli space. We define a topology on the marked moduli space and prove that it reduces to the usual Teichmüller space in case the surface is of finite type. We prove that the action of the mapping class group on this marked moduli space is continuous.

1. INTRODUCTION

The Teichmüller space of a fixed finite type surface can be thought of as the moduli space of either marked Riemann surface structures or marked complete hyperbolic structures on the surface. The two viewpoints are equivalent due to the uniformisation theorem and the fact that the isometries of the hyperbolic plane are exactly its biholomorphisms. The mapping class group of the surface acts on the Teichmüller space by change of marking. This action has been studied classically, with important consequences for the mapping class group such as the Nielsen–Thurston classification of mapping classes, the geometric classification of mapping tori, the solution to the Nielsen realisation problem, et cetera (see [Thu88], [Ber78], [FM11], [Hub06], [Hub16], [Hub22], [Thu98], [Ker83]). In this paper, we introduce the moduli space $\mathcal{T}(S)$ of marked, complete, Nielsen-convex (that is, having empty ideal boundary) hyperbolic structures on a surface S of negative Euler characteristic. We also refer to $\mathcal{T}(S)$ as the *marked moduli space*, and to a point in it as a *marked hyperbolic structure*. We are especially interested in studying the marked moduli spaces of infinite type surfaces S. In analogy with Teichmüller space, we show that the mapping class group $\mathrm{MCG}(S)$ of the surface S, a topological group, acts on the marked moduli space by change of marking. The main result of this paper is that the action of $\mathrm{MCG}(S)$ on $\mathcal{T}(S)$ is continuous. We also prove that the space $\mathcal{T}(S)$ reduces to the usual Teichmüller space in case S is a finite type surface.

In fact, the Teichmüller space of a surface has already been defined and studied even when the surface is of infinite type, and it dates back to the early days of Teichmüller theory (see [Ber63, p333] or [Ber64, §1.3]). However, it is defined for a Riemann surface X rather than its underlying topological surface S (whether of finite or infinite type), and is known

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as the quasiconformal Teichmüller space, denoted $T_{qc}(X)$. In particular, the Teichmüller space depends on the quasiconformal class of the 'basepoint' Riemann surface structure on S, namely the one provided by X. Consequently, the quasiconformal mapping class group $\mathrm{QMCG}(X)$, consisting of quasiconformal self homeomorphisms of the Riemann surface X modulo homotopy relative to its ideal boundary, acts on $T_{qc}(X)$ by change of marking. Now the relationship between $\mathrm{QMCG}(X)$ and $\mathrm{MCG}(S)$ is complicated since the ideal boundary is considered in one and disregarded in the other, but is nice in the absence of ideal boundary. In general, there is the obvious group morphism from $\mathrm{QMCG}(X)$ to $\mathrm{MCG}(S)$ obtained by forgetting the quasiconformality of the homeomorphism. If X is Nielsen-convex, this group morphism is injective, and so $\mathrm{QMCG}(X)$ is a subgroup of $\mathrm{MCG}(S)$. However, in case the surface is of infinite type, $\mathrm{QMCG}(X)$ is a proper subgroup of $\mathrm{MCG}(S)$, and $\mathrm{MCG}(S)$ does not act on $T_{qc}(X)$ by change of marking. On the other hand, the $\mathrm{MCG}(S)$ does act on $\mathcal{T}(S)$, so we expect $\mathcal{T}(S)$ to be a useful space for studying $\mathrm{MCG}(S)$. Note that if the surface is of finite type, there is only one quasiconformal class of Nielsen-convex Riemann surfaces, and therefore only one Teichmüller space associated with the topological surface; the quasiconformal mapping class group coincides with the mapping class group, which therefore acts on the Teichmüller space, leading to a rich theory. Another reason to consider $\mathcal{T}(S)$ is a theorem of Thurston ([Thu86, Corollary 5.4]) that there exists an essentially unique earthquake between any two relative hyperbolic structures on a complete hyperbolic surface. In fact, our definition of $\mathcal{T}(S)$ is inspired by this paper.

Organisation of the paper: In Section 2, we define the marked moduli space as a set, and state the main results of this paper. In Section 3, we fix notation and recall some basic facts of hyperbolic geometry and algebraic topology. In Section 4, we prove a key proposition about homeomorphisms at infinity, a tool that will be used repeatedly. In Section 5, we define the topology on the marked moduli space using homeomorphisms at infinity. In Section 6, we prove that the action of the mapping class group on the marked moduli space is continuous. In Section 7, we prove that the topology on the marked moduli space agrees with the topology coming from injecting it into the $\mathrm{PSL}(2,\mathbb{R})$-character space of the fundamental group. It follows that the topology of the marked moduli space reduces to the usual topology of the Teichmüller space in case the surface is of finite type.

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2. Definitions and Statement of Results

As in the introduction, fix a connected, oriented surface S with negative Euler characteristic, or equivalently, a nonabelian fundamental group. The Euler characteristic need not be finite; as mentioned in the introduction, our emphasis is on *infinite type surfaces*, those whose Euler characteristic is $\chi(S) = -\infty$, or equivalently, surfaces whose fundamental group is not finitely generated. In this paper, a surface is a two dimensional manifold, connected and without boundary unless otherwise specified. All surfaces in this paper are oriented and all homeomorphisms between surfaces (including isometries between hyperbolic surfaces) are orientation preserving; all self homeomorphisms of the circle are also orientation preserving, and we suppress mention of orientation. We now define the set $\mathcal{T}(S)$.

Definition 2.1 (The set underlying the marked moduli space)**.**

$$\mathcal{T}(S) := \left\{ (X, f) \middle| \begin{array}{l} X \textit{ is a complete Nielsen-convex hyperbolic surface} \\ f\colon S \to X \textit{ is a homeomorphism} \end{array} \right\} \Big/ \sim \tag{1}$$

where $(X_1, f_1) \sim (X_2, f_2)$ *if there is an isometry* $\varphi\colon X_1 \to X_2$ *homotopic to* $f_2 \circ f_1^{-1}$.

As is standard in finite type Teichmüller theory, we call the homeomorphism f the *marking map.* Here and henceforth in this paper, the phrase 'complete' hyperbolic surface always means a 'geodesically complete' hyperbolic surface. The adjective *Nielsen-convex* applied to a complete hyperbolic surface is discussed and characterised in Section 3.4 and Proposition 3.1. One should think of a complete Nielsen-convex hyperbolic surface as one which has empty 'ideal boundary', recalled in Section 3.2.

The next proposition says that the theory of the marked moduli space is not a trivial theory.

Proposition 2.2. *The set* $\mathcal{T}(S)$ *is nonempty.*

Proof. Take a topological pants decomposition $\mathcal{P}$ of S, where a pant is a surface of zero genus, b boundary components and n punctures with $b + n = 3$. Such a decomposition exists since $\chi(S) < 0$. For each pant in the scheme $\mathcal{P}$, consider a hyperbolic pair of pants with geodesic boundary, whose components are called *cuffs*, and with cusps at punctures. The geometry of the hyperbolic pairs of pants is chosen so that the lengths of any two cuffs that get glued in the scheme $\mathcal{P}$ are equal. This allows the hyperbolic pairs of pants to be glued according to the scheme $\mathcal{P}$ to produce a hyperbolic surface X homeomorphic to S via a marking map $f\colon S \to X$. If the cuff lengths are all bounded above as well as below by two fixed finite positive numbers, then the injectivity radius of points in X is bounded below by a fixed positive number, and therefore X is complete. Further, since X is a union of hyperbolic pairs of pants, Proposition 3.1 asserts that X is a complete, Nielsen-convex hyperbolic surface. Therefore $[X, f] \in \mathcal{T}(S)$, and so $\mathcal{T}(S)$ is nonempty. □

Remark 2.3. *Even if the cuff lengths are arbitrary,* [Ba19, Theorem 5.1] *asserts that the hyperbolic pairs of pants may be glued with particular choices of twists in such a way that the resulting surface* X *is geodesically complete.*

Observe that if S is a closed surface, then $\mathcal{T}(S)$ as above reduces, as a set, to the usual Teichmüller space. This is because a closed hyperbolic surface has empty ideal boundary, and does not have visible ends (or any ends at all). This observation holds true for finite type surfaces also. Indeed, if S is a finite type surface, then in the usual definition of Teichmüller space, the hyperbolic surface X is constrained to have finite area. Then each end of X has cusped geometry and so is not a visible end, which means that X has empty ideal boundary. Thus $\mathcal{T}(S)$ is an extension of the usual Teichmüller space to infinite type surfaces, as a set. In Corollary 7.6, we strengthen this observation by showing that $\mathcal{T}(S)$ is an extension of the usual Teichmüller space as a topological space as well.

The *mapping class group* of the surface S is $\mathrm{MCG}(S) = \mathrm{Homeo}^+(S) \big/ \mathrm{Homeo}_0(S)$. The group $\mathrm{Homeo}^+(S)$ is a topological group with the compact-open topology, which agrees, for

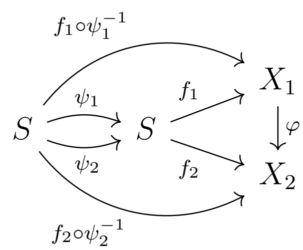


FIGURE 1. The action of $\mathrm{MCG}(S)$ on $\mathcal{T}(S)$ is well defined.

any metric on S, with the topology of uniform convergence on compact subsets. The subgroup $\mathrm{Homeo}_0(S)$ of homeomorphisms isotopic to the identity is a closed subgroup. Hence the quotient $\mathrm{MCG}(S)$ is a topological group with the quotient topology (see [AV20, Section 2.3]). It is clear that $\mathrm{MCG}(S)$ acts on $\mathcal{T}(S)$ by change of marking. We denote the action of the mapping class $[\psi]$ by the function $A_{[\psi]}$ from $\mathcal{T}(S)$ to itself. The action is defined precisely in equation (2) below.

Proposition 2.4. *There is a well defined group action* $A\colon \mathrm{MCG}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$*, currying which yields, for every mapping class* $[\psi] \in \mathrm{MCG}(S)$*, a function* $A_{[\psi]}\colon \mathcal{T}(S) \to \mathcal{T}(S)$*, defined by*

$$A_{[\psi]}[X, f] = A([\psi], [X, f]) \coloneqq [X, f \circ \psi^{-1}] \tag{2}$$

Proof. We need to show that $A([\psi], [X, f])$ is independent of two choices made in the definition of A, namely, the choice of the representative (X, f) of the marked hyperbolic structure, and the choice of the representative homeomorphism ψ of the mapping class. This follows easily from the diagram in Figure 1, which homotopy commutes. □

The main result of this paper is the following:

Theorem 2.5. *The set* $\mathcal{T}(S)$ *has a geometrically defined topology (see Definition 5.5 and Theorem 7.5), which agrees with the usual topology on the Teichmüller space when* S *is of finite type (see Corollary 7.6). With respect to this topology, the action function* A *is continuous (see Theorem 6.6) and* $\mathrm{MCG}(S)$ *acts on* $\mathcal{T}(S)$ *by homeomorphisms (see Corollary 6.7).*

Having defined the topology of the marked moduli space, it is natural to ask what geometry it admits. In particular, we seek a metric on the marked moduli space so that the mapping class group acts isometrically on the marked moduli space.

Question 2.6. *Is there a natural (geometrically defined)* $\mathrm{MCG}(S)$*-invariant metric on* $\mathcal{T}(S)$*?*

3. Notation and Preliminaries

In this section, we recall some basic facts of hyperbolic geometry and algebraic topology, allowing us to fix notation of various objects used throughout the paper. The reader familiar with these subjects may well skip Sections 3.1 to 3.5. In Section 3.4, we include a discussion of Nielsen-convexity, which appears in the definition of $\mathcal{T}(S)$. In Section 3.5, we carefully prove the existence of straight line homotopy, and in Section 3.6, we recall the 'Douady–Earle extension'. These will be used crucially in Section 5 while defining the topology on the marked moduli space, as well as in Section 6 while proving the continuity of the mapping class group action on the marked moduli space.

3.1. **The hyperbolic plane, the circle at infinity and the action of** $\mathrm{PSL}(2,\mathbb{R})$**.** A reference for this section is [And06, Chapters 1–2]. The *hyperbolic plane* may be described as either the upper half plane or the unit disk in the Riemann sphere, and is equipped with the Poincaré metric and the usual orientation. These two models of the hyperbolic plane are isometric. We denote the hyperbolic plane by $\mathbb{H}^2$. It has a *circle at infinity* which is the real projective line in the upper half plane model or the standard unit circle in the unit disk model. We denote the circle at infinity by S^1. It inherits an orientation as the boundary of the closed disk $\mathbb{H}^2 \cup S^1$, and, being a topologically a circle, also possesses a corresponding circular order. We fix, once and for all, a positively oriented triple of distinct points on S^1, such as $(0, 1, \infty)$ in the real projective line model.

The group $\mathrm{PSL}(2,\mathbb{R})$ of 2×2 real matrices of determinant 1, modulo $\pm I$, acts by fractional linear transformations on the upper half plane. This action induces an isomorphism between $\mathrm{PSL}(2,\mathbb{R})$ and the group of (orientation preserving) isometries of $\mathbb{H}^2$. Each isometry of $\mathbb{H}^2$ extends to a (an orientation preserving) homeomorphism of S^1. In fact, this action of $\mathrm{PSL}(2,\mathbb{R})$ on the circle at infinity is simply the faithful action by linear fractional transformations on the real projective line. It induces an isomorphism between $\mathrm{PSL}(2,\mathbb{R})$ and a subgroup of $\mathrm{Homeo}^+(S^1)$, which is the group of (orientation preserving) homeomorphisms of the circle. Moreover, $\mathrm{PSL}(2,\mathbb{R})$ acts freely and transitively on positively oriented triples of distinct points in S^1. For a positively oriented triple (a, b, c) of distinct points in S^1, we denote by $M(a, b, c)$ the unique element of $\mathrm{PSL}(2,\mathbb{R})$ that maps 0 to a, 1 to b and ∞ to c; The function $(a, b, c) \mapsto M(a, b, c)$ is continuous. In particular, the faithfulness of this action means that an isometry of $\mathbb{H}^2$ is determined by its induced homeomorphism of the circle at infinity. We will freely use the same symbol to denote an element of $\mathrm{PSL}(2,\mathbb{R})$ whether it is viewed as an isometry of $\mathbb{H}^2$ or a homeomorphism of S^1.

A non-trivial element of $\mathrm{PSL}(2,\mathbb{R})$ is one of exactly three types depending on the number of points it fixes in $\mathbb{H}^2 \cup S^1$, namely, *elliptic* (exactly one fixed point, which is in $\mathbb{H}^2$), *parabolic* (exactly one fixed point, which is in S^1), or *hyperbolic* (exactly two fixed points, both in S^1). For a hyperbolic γ, we denote by γ_∞, the unique attracting fixed point of γ on S^1, also known as its *sink*; the function $\gamma \mapsto \gamma_\infty$ is continuous. Then $(\gamma^{-1})_\infty$ is the unique repelling fixed point of γ on S^1, and is called its *source*. Further, note that $\mathrm{Homeo}^+(S^1)$ is a topological group with the compact-open topology, which agrees, for any metric on S^1, with the topology of uniform convergence. Then $\mathrm{PSL}(2,\mathbb{R})$ is a closed and embedded subgroup of $\mathrm{Homeo}^+(S^1)$.

3.2. **Complete hyperbolic surfaces and Fuchsian groups.** A reference for this section is [Kat92, Chapters 2–3]. Let X be a complete (and connected and oriented) hyperbolic surface. We denote by d_X the *distance function* on X, and by inj_X the *injectivity radius* as a function on X, which is well known to be a continuous function. Being a complete hyperbolic surface, X has a (Riemannian locally isometric, orientation preserving) universal cover $p_X \colon \mathbb{H}^2 \to X$, and any other such cover is of the form $p_X \circ \sigma$ for some $\sigma \in \mathrm{PSL}(2,\mathbb{R})$. We denote the deck group of the universal cover p_X by Γ_X, which is a group of isometries of $\mathbb{H}^2$ acting freely and properly discontinuously on $\mathbb{H}^2$. Alternatively, as a subgroup of $\mathrm{PSL}(2,\mathbb{R})$, Γ_X is a torsion-free discrete group, also known as a torsion-free *Fuchsian* group. The universal cover induces an isometry between $\Gamma_X \backslash \mathbb{H}^2$ and X. For any $x \in X$, $\mathrm{inj}_X(x)$ is the radius of the largest disk centred at x which is evenly covered by the universal cover. Thus if $d_X(x, y) < \mathrm{inj}_X(x)$, then there is a unique geodesic segment from x to y.

The deck group Γ_X, being a subgroup of $\mathrm{PSL}(2,\mathbb{R})$, acts on the circle at infinity S^1 as well. As a group acting freely and properly discontinuously by isometries on $\mathbb{H}^2$, Γ_X cannot contain any elliptic elements. Thus all the non-trivial elements of Γ_X are either hyperbolic or parabolic. The discreteness of Γ_X implies that each element of Γ_X has a root which is *primitive*, that is, not a non-trivial power of another element of Γ_X. If Γ_X is non-trivial, and not a cyclic group generated by a parabolic element, then it must have at least one hyperbolic element. Any two hyperbolic elements of Γ_X which have the same sink also have the same source, and are in fact both positive powers of some hyperbolic in Γ_X. We denote by $(\Gamma_X)_\infty$ the set of sinks of all the hyperbolic elements of Γ_X, and it is easily seen to be Γ_X-invariant.

The set of accumulation points in $\mathbb{H}^2 \cup S^1$ of the Γ_X-orbit of some (any) point in $\mathbb{H}^2$ is a closed subset of S^1, known as the *limit set* Λ_X of Γ_X. If Λ_X is nonempty, it is the smallest nonempty Γ_X-invariant closed subset of S^1. Thus Λ_X is the closure of $(\Gamma_X)_\infty$ in S^1, if $(\Gamma_X)_\infty$ is nonempty. Γ_X acts freely and properly discontinuously on $S^1 \setminus \Lambda_X$, and the quotient $\Gamma_X \backslash (S^1 \setminus \Lambda_X)$ is called the *ideal boundary* $I(X)$ of X (see [Ber63, p334] or [Ber64, §1.3] or [Hub06, Section 3.7]). The convex hull $\mathrm{CH}(\Lambda_X)$ of Λ_X in $\mathbb{H}^2$ is called the *Nielsen convex region* of Γ_X, easily seen to be Γ_X-invariant, and its quotient by Γ_X is the *convex core* $C(X)$ of X. The Fuchsian group Γ_X is said to be *of the first kind* if its limit set Λ_X is the entire circle at infinity.

3.3. **The algebraic topology of hyperbolic surfaces.** A reference for this section is [Hat02, Section 1.3]. As before, let X be a complete (and connected and oriented) hyperbolic surface. Given the universal cover $p_X\colon \mathbb{H}^2 \to X$, a basepoint $x \in X$ and a choice of its lift $\tilde{x} \in \mathbb{H}^2$, or equivalently, given a pointed universal cover $p_X\colon (\mathbb{H}^2, \tilde{x}) \to (X, x)$, there is an isomorphism $\varphi_X\colon \pi_1(X, x) \to \Gamma_X$, known as the *holonomy representation*, and it is described as follows. If α is an oriented closed curve in X based at x, then $\varphi_X[\alpha]$ is the unique deck transformation, called the *holonomy around* α, that maps $\tilde{x}$ to the endpoint of the lift of α starting at $\tilde{x}$. If $\tilde{\alpha}$ is the bi-infinite lift of α passing through $\tilde{x}$, then the deck transformation $\varphi[\alpha]$ acts on $\tilde{\alpha}$ by translation. Note that the isomorphism φ_X depends not only on the choice of the basepoint, but also on the choice of lift $\tilde{x}$, and it changes with these choices as follows. Suppose $x' \in X$ is another basepoint and $\tilde{x}' \in \mathbb{H}^2$ is a choice of its lift with the corresponding holonomy representation $\varphi'_X\colon \pi_1(X, x') \to \Gamma_X$. Then the two holonomy representations satisfy the relation $\varphi'_X = \varphi_X \circ c_\beta$. Here β is the projection $p(\tilde{\beta})$ of a continuous path $\tilde{\beta}$ from $\tilde{x}$ to $\tilde{x}'$, and c_β is 'conjugation by β', that is, $c_\beta[\alpha'] = [\beta \cdot \alpha' \cdot \overline{\beta}]$ for all $[\alpha'] \in \pi_1(X, x')$. If α, based at x, is homotopic to α', based at x', and β is the track of x under this homotopy, then $[\alpha] = c_\beta[\alpha']$. Combining these two observations, we see that $\varphi_X[\alpha] = \varphi'_X[\alpha']$. In other words, if two closed curves are homotopic, then the holonomies around them (with respect to appropriate choices of lifts of basepoints under the same universal cover) are equal. Conversely, if the holonomies around two curves are equal, then the curves are free homotopic.

If X, Y are complete hyperbolic surfaces with universal covers $p_X\colon \mathbb{H}^2 \to X$ and $p_Y\colon \mathbb{H}^2 \to Y$, then any homeomorphism $f\colon X \to Y$ lifts to a homeomorphism $\tilde{f}\colon \mathbb{H}^2 \to \mathbb{H}^2$. This induces an isomorphism $f_*\colon \Gamma_X \to \Gamma_Y$ between the two deck groups, which is 'conjugation by $\tilde{f}$', that is, $f_*(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$ for all $\gamma \in \Gamma_X$. Note that the isomorphism f_* depends on the choice of the lift $\tilde{f}$, but we denote it by f_*, suppressing the lift from the notation. We also write the above relation as $\tilde{f} \circ \gamma = f_*(\gamma) \circ \tilde{f}$ for all $\gamma \in \Gamma_X$. Collecting all these equalities together, we have an equality of sets $\tilde{f} \circ \Gamma_X = \Gamma_Y \circ \tilde{f}$. This set is the set of all lifts of f. Conversely, if $F : \mathbb{H}^2 \to \mathbb{H}^2$ satisfies $F \circ \Gamma_X = \Gamma_Y \circ F$, then F descends to a homeomorphism from X to Y. In this situation, we say that $\tilde{f}$ (or F) is *equivariant* under the action of the deck groups. If φ_X and φ_Y (with respect to basepoints $\tilde{y} = \tilde{f}(\tilde{x})$ and $y = p_Y(\tilde{y})$) are the holonomy representations of X and Y respectively, then we have $f_*(\varphi_X[\alpha]) = \varphi_Y(f_*[\alpha])$. Here the f_* on the left hand side is the isomorphism between the deck groups, whereas on the right hand side, f_* is the π_1 functor. In other words, if $\gamma \in \Gamma_X$ is the holonomy around the oriented closed $\alpha \subset X$, then $f_*(\gamma) \in \Gamma_Y$ is the holonomy around the oriented closed curve $f(\alpha)$.

3.4. **Nielsen-convexity for complete hyperbolic surfaces.** We have adopted the term *Nielsen-convex* from work of Alessandrini, Liu and others (see [ALP+11, Definition 4.3]). They actually define a notion of Nielsen-convexity for hyperbolic surfaces which may or may not be complete. However, we are interested only in complete hyperbolic surfaces. So instead of recalling their original definition, we simply characterise Nielsen-convexity for complete hyperbolic surfaces in multiple different but equivalent ways in Proposition 3.1 below. Note that in [ALP+11], the hyperbolic pair of pants with three cusps is treated separately due to technical reasons concerning their definition of Nielsen-convexity. However, since it satisfies the conditions below, we will call it Nielsen-convex. Notation from Section 3.2 is used, and the objects in some of the conditions below are defined post the choice of a pointed universal cover.

Proposition 3.1 (Nielsen-convexity)**.** *Let X be a complete hyperbolic surface. Then the following are equivalent.*

1. *X is Nielsen-convex according to* [ALP+11, Definition 4.3].
2. *The ideal boundary $I(X)$ of X is empty.*
3. *The convex core $C(X)$ of X equals X.*
4. *The limit set Λ_X of the deck group Γ_X, or equivalently, of the action of $\pi_1(X)$ on the universal cover $\mathbb{H}^2$, is the entire circle at infinity S^1.*
5. *The set $(\Gamma_X)_\infty$ of sinks of hyperbolic elements in Γ_X is dense in S^1.*
6. *X is isometric to ${}_{\Gamma_X}\backslash\mathbb{H}^2$ for some torsion-free Fuchsian group Γ_X of the first kind.*
7. *X can be constructed by gluing hyperbolic pairs of pants (possibly with cusps) along their boundary components.*
8. *X has no visible ends.*

Proof. The equivalence of the first and the third conditions is [ALP+11, Proposition 4.6] (applied to complete hyperbolic surfaces), and that of the first and the seventh conditions is part of [ALP+11, Theorem 4.5]. The equivalence of the third and the fourth conditions follows easily from the definitions in Section 3.2 above. Indeed, since $C(X) = {}_{\Gamma_X}\backslash \mathrm{CH}(\Lambda_X)$ and $X = {}_{\Gamma_X}\backslash\mathbb{H}^2$, $C(X) = X$ if and only if $\mathrm{CH}(\Lambda_X) = \mathbb{H}^2$, which is equivalent to $\Lambda_X = S^1$. The sixth condition is really just a restatement of the fourth, and the equivalence of the second and the fourth conditions also follows straight from definitions. To see the equivalence of the fourth and the fifth conditions, first observe that neither the hyperbolic plane nor its quotient by a single parabolic satisfy either of the two conditions. Thus Γ_X contains at least one hyperbolic. The set $(\Gamma_X)_\infty$ is nonempty, and hence dense in Λ_X. Thus $\Lambda_X = S^1$ if and only if $(\Gamma_X)_\infty$ is dense in S^1, establishing the equivalence. For a discussion of visible ends, see [Ba19, Section 2]. The equivalence of the third and the eighth conditions above follows easily from Lemma 2.2 there. □

3.5. **Straight line homotopy.** We will have occasion to use a straight line homotopy between two maps from a topological space into a hyperbolic surface, and we explain this term in this section. We provide a detailed proof of the facts that any two maps to a hyperbolic surface are connected by a unique straight line homotopy if they are sufficiently close.

Definition 3.2 (Straight line homotopy)**.** *Let Z be a topological space, X a complete hyperbolic surface and $h_0, h_1 \colon Z \to X$ be continuous functions. We say that a homotopy h_t between h_0 and h_1 is a* straight line homotopy *if for every $q \in Z$, there is a unique shortest geodesic segment in X from $h_0(q)$ to $h_1(q)$, and that $t \mapsto h_t(q)$ is a constant speed parametrisation of this geodesic segment.*

It is clear that if a straight line homotopy exists between two maps, then it is unique, although this uniqueness will not have any consequences for our purpose. We are concerned only with the existence of the straight line homotopy. The main thrust of the proposition below is that $h_t(q)$ as described above is actually jointly continuous in q and t, that is, the function $h \colon Z \times [0,1] \to X$ given by $h(q,t) \coloneqq h_t(q)$ is continuous with respect to the product topology on $Z \times [0,1]$.

Proposition 3.3 (Straight line homotopy on hyperbolic surfaces)**.** *Use the notation of Definition 3.2, and assume that*

$$\text{for each } q \in Z, \text{ we have } d_X(h_0(q), h_1(q)) < \operatorname{inj}_X(h_0(q)) \tag{3}$$

Then there is a straight line homotopy between h_0 and h_1.

Proof. For each $q \in Z$, since $h_1(q)$ is contained in the disk centred at $h_0(q)$ of radius equal to its injectivity radius, there is a unique geodesic segment joining $h_0(q)$ to $h_1(q)$ (possibly degenerate of of zero length, a degeneracy that occurs exactly when $h_1(q)$ coincides with $h_0(q)$). So we can define $h_t(q)$ by declaring that $t \mapsto h_t(q)$ parametrises this geodesic segment with constant speed $d_X(h_0(q), h_1(q))$. It remains to be shown that the function h is continuous.

First we prove continuity in the case when the hyperbolic surface is $\mathbb{H}^2$. Note that the injectivity radius at any point on the hyperbolic plane is infinite, so the inequality (3) is always satisfied for any Z, h_0, h_1. let $T\mathbb{H}^2$ be the tangent bundle of $\mathbb{H}^2$ and $\pi \colon T\mathbb{H}^2 \to \mathbb{H}^2$ be the natural projection. The function $\Theta \colon T\mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2$ given by $\Theta(v) = (\pi(v), \exp(v))$ is continuous because of continuous dependence of solutions of ordinary differential equations on initial values, and bijective because there is a unique geodesic segment between any pair of points on $\mathbb{H}^2$. Thus Θ is a continuous bijection between manifolds of the same dimension, and hence is a homeomorphism. Then the required straight line homotopy is $h_t(q) = \exp(t\Theta^{-1}(h_0(q), h_1(q)))$, and h is clearly continuous.

Now we tackle the case of an arbitrary hyperbolic surface X. Suppose that $p_X \colon \mathbb{H}^2 \to X$ is a universal cover, and let $q_0 \in Z$ be an arbitrary point. We construct an open neighbourhood U of q_0 in Z as follows, and then show that h is continuous over $U \times [0,1]$. The disk B of radius $\operatorname{inj}_X(h_0(q_0))$ centred at $h_0(q_0)$ is evenly covered by the universal cover, so the function p_X has a local inverse $P \colon B \hookrightarrow\!\!\!\!\!\rightarrow \tilde{B}$. That is, P maps B diffeomorphically onto a disk $\tilde{B} \subset \mathbb{H}^2$. Now p_X and P are Riemannian isometries, so they map Riemannian geodesics to Riemannian geodesics. However, note that p_X and P are not necessarily metric isometries; they may not preserve the restriction to B of the distance function d_X.

Let U be the set of all points $q \in Z$ such that $h_0(q), h_1(q) \in B$ and $d_{\mathbb{H}^2}(P \circ h_0(q), P \circ h_1(q)) < \operatorname{inj}_X(h_0(q))$, which contains q_0 due to hypothesis and the construction of B and P. To show that U is open, consider the sets $U_0 = h_0^{-1}B$ and $U_1 = h_1^{-1}B$, which are open because h_0 and h_1 are continuous functions. Therefore $P \circ h_0$ and $P \circ h_1$ are functions defined on $U_0 \cap U_1$, and are continuous on this domain of definition. Thus the real valued function $\eta(q) = \operatorname{inj}_X(h_0(q)) - d_{\mathbb{H}^2}(P \circ h_0(q), P \circ h_1(q))$ is defined and continuous on $U_0 \cap U_1$, and the preimage $U = \eta^{-1}(0, +\infty)$ is an open set in Z. Since $q_0 \in U$ as noted above, U is an open neighbourhood of q_0 in Z.

Next, let $q \in U$ be arbitrary. There is a unique geodesic segment γ from $P \circ h_0(q)$ to $P \circ h_1(q)$ in $\tilde{B} \subset \mathbb{H}^2$, parametrised by $\gamma(t) = \exp(t\Theta^{-1}(P \circ h_0(q), P \circ h_1(q)))$ as above. Note that here the parameter t takes values in $[0, 1]$, and the parametrisation is at constant speed equal to the length $l(\gamma)$ of γ. Now the length $l(\gamma)$ is equal to the hyperbolic distance between the two endpoints $P \circ h_0(q)$ and $P \circ h_1(q)$, and is less than $\operatorname{inj}_X(h_0(q))$ due to the definition of U. Since p_X is a Riemannian isometry and its inverse is P, p_X maps γ to a geodesic segment $p_X \circ \gamma$ from $h_0(q)$ to $h_1(q)$ in X, which has the same length $l(p_X \circ \gamma) = l(\gamma)$, and which is also parametrised at the same constant speed $l(\gamma)$. Since $l(p_X \circ \gamma) = l(\gamma) < \operatorname{inj}_X(h_0(q))$, we conclude that $p_X \circ \gamma$ is in fact the unique shortest geodesic segment from $h_0(q)$ to $h_1(q)$ in X, and so its length $l(p_X \circ \gamma)$ equals the distance $d_X(h_0(q), h_1(q))$. But we defined h_t by declaring $t \mapsto h_t(q)$ to be the parametrisation of this unique shortest geodesic segment at constant speed $d_X(h_0(q), h_1(q))$. Therefore

$$h(q, t) = h_t(q) = p_X \circ \exp(t\Theta^{-1}(P \circ h_0(q), P \circ h_1(q))) \tag{4}$$

As $q \in U$ and $t \in [0, 1]$ are arbitrary, this expression of $h_t(q)$ holds for all $(q, t) \in U \times [0, 1]$.

Thus we have found an expression for h over the domain $U \times [0, 1]$ which is built by composing continuous functions, homeomorphisms, diffeomorphisms, scalar multiplication in finite dimensional vector spaces, and functions into product spaces whose components are continuous. By standard point set topology, h is continuous over $U \times [0, 1]$, and in particular continuous at all points in $\{q_0\} \times [0, 1] \subset Z \times [0, 1]$. Since $q_0 \in Z$ was arbitrary, h is continuous over all of $Z \times [0, 1]$, and therefore defines a straight line homotopy as claimed. ☐

Remark 3.4. *Proposition 3.3 holds for arbitrary Riemannian manifolds X if quantified suitably stringently. That is, in the inequality* (3)*, $\operatorname{inj}_X(h_0(q))$ should be replaced by a suitable smaller quantity.*

3.6. **The Douady–Earle extension.** In this section, we recall the *Douady–Earle extension* from [DE86]. The Douady–Earle extension is a construction that extends homeomorphisms of S^1 to homeomorphisms of $\mathbb{H}^2 \cup S^1$ in a conformally natural way. This means that the extension of an element of $\mathrm{PSL}(2, \mathbb{R})$ as a homeomorphism of S^1 is the same element, now viewed as a homeomorphism of $\mathbb{H}^2$; and that the extension respects composition on the left or right side by elements of $\mathrm{PSL}(2, \mathbb{R})$. However, note that this construction does not yield a group homomorphism. More precisely,

Theorem 3.5 (see [DE86, Sections 3–4])**.** *There is a function, 'the Douady–Earle extension'* $\mathrm{DE}\colon \mathrm{Homeo}^+(S^1) \to \mathrm{Homeo}^+(\mathbb{H}^2 \cup S^1)$ *such that all the following hold:*

(1) For every $f \in \mathrm{Homeo}^+(S^1)$, we have $\mathrm{DE}(f)|_{S^1} = f$. Thus $\mathrm{DE}(f)$ extends f.
(2) $\mathrm{DE}(\mathrm{id}_{S^1}) = \mathrm{id}_{\mathbb{H}^2 \cup S^1}$.
(3) (Conformal naturality) *For every $f \in \mathrm{Homeo}^+(S^1)$ and every $\sigma_1, \sigma_2 \in \mathrm{PSL}(2, \mathbb{R})$, we have $\mathrm{DE}(\sigma_1 \circ f \circ \sigma_2) = \sigma_1 \circ \mathrm{DE}(f) \circ \sigma_2$.*

(4) DE *is continuous (when* $\mathrm{Homeo}^+(\mathbb{H}^2 \cup S^1)$ *is given the compact-open topology, or equivalently, the topology of uniform convergence on compact sets).*

The results proved in [DE86] are significantly stronger, but the above statement is sufficient for our purpose.

4. A Key Tool: The Homeomorphism at Infinity

In this section, we provide a detailed proof of the Proposition 4.1 (Homeomorphism at infinity), which is actually a special case of [Thu86, Proposition 5.3]. In that paper, Thurston used it to reduce the earthquake theorem for general hyperbolic surfaces to a version of the earthquake theorem for their universal covers (that is, the hyperbolic plane). We will use homeomorphisms at infinity repeatedly in the rest of the paper.

Proposition 4.1 (Homeomorphism at infinity)**.** *Let X, Y be complete hyperbolic surfaces. Fix universal covers $p_X : \mathbb{H}^2 \to X$, $p_Y : \mathbb{H}^2 \to Y$, and let Γ_X, Γ_Y be the respective deck groups. Let $f : X \to Y$ be a homeomorphism with a lift $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$ to the universal covers, which induces an isomorphism $f_* : \Gamma_X \to \Gamma_Y$ of deck groups. Assume that both X and Y are Nielsen-convex. Then*

(1) The lift $\tilde{f}$ extends to a homeomorphism at infinity $\partial\tilde{f} : S^1 \to S^1$.
(2) The function $\hat{f}$ which equals $\tilde{f}$ on $\mathbb{H}^2$ and $\partial\tilde{f}$ on S^1 is a homeomorphism of $\mathbb{H}^2 \cup S^1$.
(3) The homeomorphism $\partial\tilde{f}$ is equivariant under the action of the deck groups. That is, for every $\gamma \in \Gamma_X$, we have $(\partial\tilde{f}) \circ \gamma = f_*(\gamma) \circ (\partial\tilde{f})$. Thus in $\mathrm{Homeo}^+(S^1)$*, the left coset $(\partial\tilde{f}) \circ \Gamma_X$ equals the right coset $\Gamma_Y \circ (\partial\tilde{f})$.*
(4) If f_t is an isotopy which lifts to an isotopy $\tilde{f}_t$, then $\partial\tilde{f}_0 = \partial\tilde{f}_1$.
(5) Suppose Z is another complete, Nielsen-convex hyperbolic surface with a universal cover $p_Z : \mathbb{H}^2 \to Z$, and $g : Y \to Z$ is a homeomorphism with lift $\tilde{g}$. Then $\partial(\tilde{g} \circ \tilde{f}) = (\partial\tilde{g}) \circ (\partial\tilde{f})$.
(6) If $\sigma \in \mathrm{PSL}(2, \mathbb{R})$, then $\sigma \circ \tilde{f}$ also extends to a homeomorphism at infinity, and $\partial(\sigma \circ \tilde{f}) = \sigma \circ (\partial\tilde{f})$.

Remark 4.2. *Note that if X, Y are closed surfaces, then $\tilde{f}$ is a quasi-isometry and so extends to a (quasi-symmetric) homeomorphism of S^1. However, for infinite type surfaces X, Y, $\tilde{f}$ may not be a quasi-isometry. The proposition asserts that nevertheless, it extends to a (not necessarily quasi-symmetric) homeomorphism of S^1.*

Remark 4.3. *Note that the homeomorphism $\partial\tilde{f}$ depends on the choice of the lift $\tilde{f}$. The other choices of the lift of f are $\gamma \circ \tilde{f}$ for $\gamma \in \Gamma_Y$ or $\tilde{f} \circ \gamma$ for $\gamma \in \Gamma_X$, leading to the homeomorphism at infinity $\gamma \circ (\partial\tilde{f})$ or $(\partial\tilde{f}) \circ \gamma$ by part (6) of the proposition.*

The outline of the proof is as follows. First, in Lemma 4.4, we prove that f_* preserves the type (hyperbolic or parabolic) of elements of the deck group. Next, in Definition 4.5, we define $\partial\tilde{f}$ on the set $(\Gamma_X)_\infty$ of sinks of hyperbolic elements of Γ_X, which is dense in S^1, and give an alternate characterisation in Lemma 4.7. Then, in Lemma 4.8, we show that $\partial\tilde{f}$ is monotonic, that is, it preserves that circular order of points in S^1. In Lemma 4.9 we show that $\partial\tilde{f}$ has no jump discontinuities, allowing us to extend it to a unique homeomorphism of S^1. Finally, in Lemma 4.10, we show that the extension yields a homeomorphism of $\mathbb{H}^2 \cup S^1$.

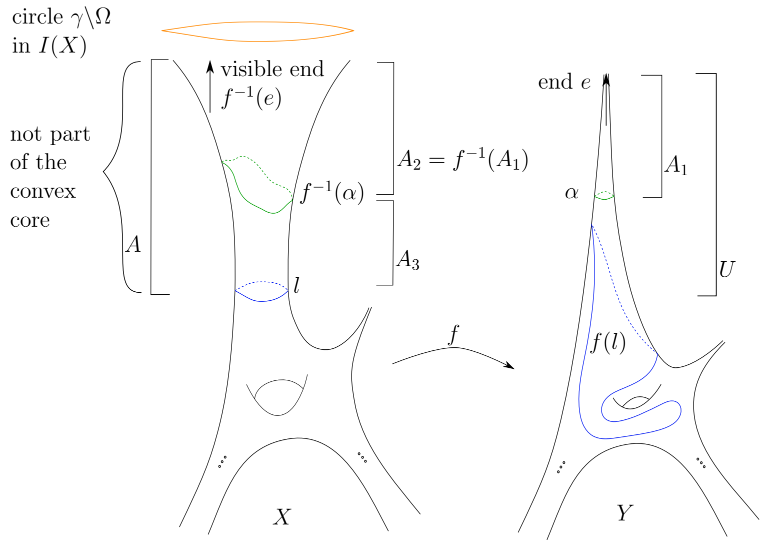


FIGURE 2. f_* is a type preserving isomorphism.

Lemma 4.4. *Keep the hypotheses of Proposition 4.1. Then f_* is a type preserving isomorphism. That is, for every $\gamma \in \Gamma_X$, $f_*(\gamma) \in \Gamma_Y$ is hyperbolic (resp. parabolic) if γ is hyperbolic (resp. parabolic).*

Proof. It is enough to prove that f_* preserves the type of primitive elements, since powers of hyperbolic and parabolic elements of $\mathrm{PSL}(2,\mathbb{R})$ are hyperbolic and parabolic respectively, and every element of Γ_X and Γ_Y has a root which is a primitive element. Suppose, for the sake of contradiction, that there is a primitive element $\gamma \in \Gamma_X$ such that γ and $f_*(\gamma)$ are not of the same type. Note that group isomorphisms map primitive elements to primitive elements and so $f_*(\gamma)$ is also a primitive element in Γ_Y. As the torsion-free Fuchsian groups Γ_X and Γ_Y cannot have elliptics, one of γ and $f_*(\gamma)$ is hyperbolic and the other is parabolic. Replacing f with f^{-1} if necessary, assume that γ is hyperbolic and $f_*(\gamma)$ is parabolic. The idea of the proof here is that the parabolic $f_*(\gamma)$ is the holonomy around a simple closed curve which is peripheral, that is, one of the components of its complement is topologically an annulus. Since f is a homeomorphism, the same then holds for the hyperbolic γ, which contradicts the Nielsen-convexity of X. We provide details below, carefully constructing these annuli.

There is a horodisk in $\mathbb{H}^2$ about the fixed point of $f_*(\gamma)$ which projects, under p_Y, to a cusp neighbourhood U of an end e of Y. See Figure 2. Let $L \subset \mathbb{H}^2$ be the axis of the hyperbolic element γ and $l \subset X$ be its projection under p_X. Then l is a closed geodesic of X and hence compact. Thus its image $f(l)$ is also compact. Hence there exists a horocyclic oriented closed curve $\alpha \subset U \subset Y$, in a sufficiently small neighbourhood of the end e, that is disjoint from $f(l)$, and such that the holonomy around α is $f_*(\gamma)$. Since $f_*(\gamma)$ is a primitive element, α is a simple closed curve. Further, α cuts Y into two components, one of which is (topologically) an annulus $A_1 \subset U$, and the other contains $f(l)$. Since f is a homeomorphism, $f^{-1}(\alpha)$ cuts X into two components, one of which is the annulus $A_2 = f^{-1}(A_1)$, and the other contains l.

Now l and $f^{-1}(\alpha)$ are closed curves in the same free homotopy class, since the holonomy around each is γ. The geodesic representative in this class is l, which is a simple curve because $f^{-1}(\alpha)$ is also a simple curve. Further, l and $f^{-1}(\alpha)$ are disjoint because $f(l)$ and

α are disjoint. Thus l and $f^{-1}(\alpha)$ are disjoint and isotopic, and hence bound an annulus $A_3 \subset X \setminus A_2$. Thus l cuts X into two components, one of which is $A = A_2 \cup f^{-1}(\alpha) \cup A_3$, an annulus. The closure of A is a complete hyperbolic surface with geodesic boundary l. Topologically it is an annulus with one boundary component, and thus is a hyperbolic funnel. That is, it is the quotient of the half plane in $\mathbb{H}^2$ bounded by L by the isometry γ translating along L. But then the cyclic subgroup generated by γ acts freely and properly discontinuously on the arc Ω of the circle at infinity cut out by L, and elements of Γ_X outside this subgroup move Ω off of itself. Therefore Ω is contained in the complement of the limit set Λ_X, and its quotient by γ is a circle. Hence the ideal boundary $I(X)$ is nonempty, and contains a circle component $\gamma\backslash\Omega$. Another way to view this situation is that L is part of the boundary of the convex hull $\mathrm{CH}(\Lambda_X)$, and therefore l is part of the boundary of the convex core of X. Hence A is not part of the convex core. Due to Proposition 3.1, this contradicts the hypothesis that X is a complete, Nielsen-convex hyperbolic surface. We conclude that f_*, the induced map between deck groups, is type preserving. □

Let $(\Gamma_X)_\infty$ and $(\Gamma_Y)_\infty$ be the sets of sinks of all the hyperbolic elements of Γ_X and Γ_Y respectively. Now we define $\partial\tilde{f}$ on $(\Gamma_X)_\infty$.

Definition 4.5. *Suppose $q \in (\Gamma_X)_\infty$, and $\gamma \in \Gamma_X$ is a hyperbolic element such that $q = \gamma_\infty$. By Lemma 4.4, f_* is type preserving, so $f_*(\gamma)$ is also a hyperbolic element. We define $\partial\tilde{f}(q)$ to be the sink $(f_*(\gamma))_\infty$.*

Lemma 4.6. *For each $q \in (\Gamma_X)_\infty$, $\partial\tilde{f}(q)$ is a well defined element of $(\Gamma_Y)_\infty$. Thus the definition above yields a well defined function $\partial\tilde{f}\colon (\Gamma_X)_\infty \to (\Gamma_Y)_\infty$, and it is a bijection.*

Proof. We need to show that $\partial\tilde{f}(q)$ is independent of the choice of the hyperbolic element $\gamma \in \Gamma_X$ in the definition of $\partial\tilde{f}$. Let γ' be another hyperbolic element in the Fuchsian group Γ_X such that $\gamma'_\infty = q$. Then γ, γ' are in fact positive powers of some hyperbolic element $\gamma'' \in \Gamma_X$. Thus $f_*(\gamma)$ and $f_*(\gamma')$ are positive powers of $f_*(\gamma'')$ and hence all three have the same sink. Thus $\partial\tilde{f}(q)$ is unambiguously defined and is a sink of a hyperbolic element of Γ_Y, so lies in $(\Gamma_Y)_\infty$. For the second assertion, note that $\partial(\tilde{f}^{-1})$ is an inverse, and hence the function $\partial\tilde{f}$ so defined is a bijection from $(\Gamma_X)_\infty$ to $(\Gamma_Y)_\infty$. □

There is another way of viewing the function $\partial\tilde{f}$ (so far defined on the subset $(\Gamma_X)_\infty$) as follows. If L is an oriented geodesic line whose forward endpoint is $q \in (\Gamma_X)_\infty$, then $\partial\tilde{f}(q)$ is the forward endpoint of $\tilde{f}(L)$.

Lemma 4.7. *Keep the hypotheses of Proposition 4.1. Let γ_∞ be the sink of a hyperbolic element $\gamma \in \Gamma_X$, and suppose L is an oriented geodesic line in $\mathbb{H}^2$ whose forward endpoint on the circle at infinity is γ_∞. Then the oriented curve $\tilde{f}(L)$ has forward endpoint $(f_*(\gamma))_\infty$ on the circle at infinity.*

Proof. Let L' be the axis of γ, oriented from its source $(\gamma^{-1})_\infty$ to its sink γ_∞. First we prove the lemma for the oriented geodesic line L'. To do so, note that γ acts by translation along L'. Hence $f_*(\gamma)$ acts by translation along $\tilde{f}(L')$. Now the images of any point in $\mathbb{H}^2$ under larger and larger positive powers of $f_*(\gamma)$ limit to its attracting fixed point $(f_*(\gamma))_\infty$. Therefore the forward endpoint on the circle at infinity of $\tilde{f}(L')$ is the sink $(f_*(\gamma))_\infty$.

Now we tackle the case of an arbitrary oriented geodesic line L with forward endpoint γ_∞. The idea of the proof here is that L is asymptotic to L', and thus the projection $p_X(L)$ is

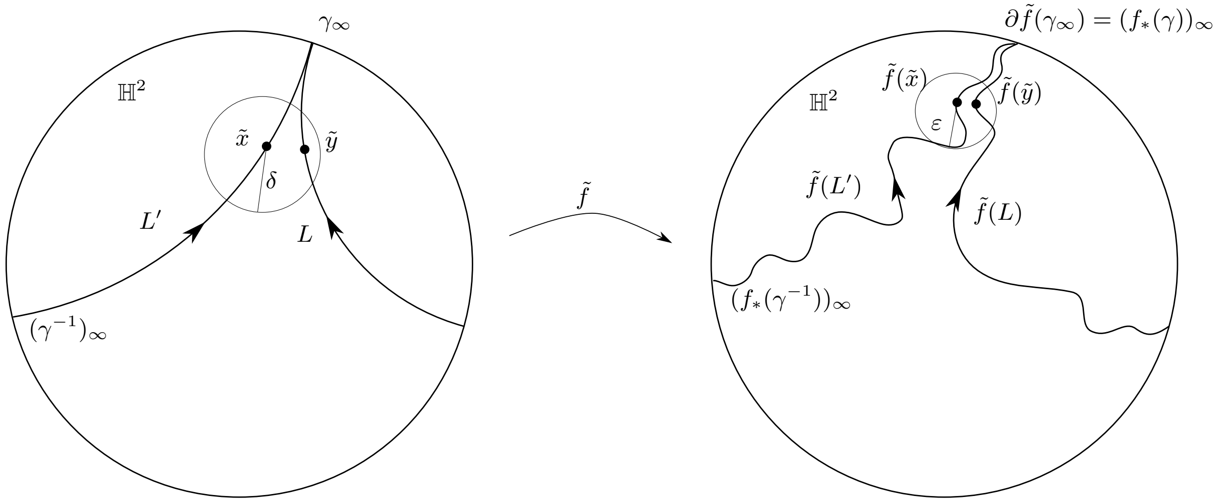


FIGURE 3. The forward endpoint of $\tilde{f}(L)$ is $(f_*(\gamma))_\infty$.

eventually a bounded distance from the closed curve $p_X(L')$. Due to uniform continuity of f on compact sets, actually the lift $\tilde{f}(L)$ of $f(p_X(L))$ is also eventually at a bounded distance from the lift $\tilde{f}(L')$ of $f(p_X(L'))$. Thus $\tilde{f}(L)$ has the same forward endpoint as $\tilde{f}(L')$, which we have shown above to be $(f_*(\gamma))_\infty$. We provide the details below, carefully lifting $f(p_X(L))$ and $f(p_X(L'))$ to $\tilde{f}(L)$ and $\tilde{f}(L')$ respectively.

Let $\tilde{K}$ denote the closed 1-neighbourhood of the axis L' in $\mathbb{H}^2$, that is, the set of points of distance at most 1 from L'. Let $K = p_X(\tilde{K})$, so that K is the closed 1-neighbourhood of the closed geodesic $p_X(L')$ in X. The set K is compact, and so is its image $f(K) \subset Y$. Since injectivity radius is a continuous function on any Riemannian manifold, the injectivity radius of points in $f(K)$ is bounded below by a number $\varepsilon > 0$. Also since K is compact, $f|_K$ is uniformly continuous. Let $\delta > 0$ be such that for all $x, y \in K$, $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$. Reducing δ if necessary, assume that $\delta \leq 1$. Since L and L' are oriented geodesic lines with the same forward endpoint, they are asymptotic. Let L_1 be a subray of L that is within the δ-neighbourhood of L'.

Take an arbitrary point $\tilde{y} \in L_1$. There exists a point $\tilde{x} \in L'$ such that $d_{\mathbb{H}^2}(\tilde{x}, \tilde{y}) < \delta$. Since $\delta \leq 1$, we have $d_{\mathbb{H}^2}(\tilde{x}, \tilde{y}) < 1$ and so $\tilde{x}, \tilde{y} \in \tilde{K}$. See Figure 3. Let α be the geodesic segment from $\tilde{x}$ to $\tilde{y}$, and let $x = p_X(\tilde{x})$, $y = p_X(\tilde{y})$. Since p_X is a Riemannian local isometry, $d_X(x, y) \leq d_{\mathbb{H}^2}(\tilde{x}, \tilde{y}) < \delta$. Hence by uniform continuity as above, $d_Y(f(x), f(y)) < \varepsilon$. In fact, the diameter of $p_X(\alpha)$ is less than δ so $f(p_X(\alpha))$ is a curve that lies entirely in the ε-neighbourhood of the point $f(x)$ in Y. Since the injectivity radius at $f(x)$ is at least ε, this ε-neighbourhood of $f(x)$ in Y is evenly covered by the universal cover p_Y, and one component of its p_Y-preimage is the ε-neighbourhood of $\tilde{f}(\tilde{x})$ in $\mathbb{H}^2$. Hence $f(p_X(\alpha))$ lifts to a curve contained in this ε-neighbourhood of $\tilde{f}(\tilde{x})$. But the endpoint of this lifted curve is $\tilde{f}(\tilde{y})$, so $d_{\mathbb{H}^2}(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{y})) < \varepsilon$. But $\tilde{f}(\tilde{y})$ is an arbitrary point on $\tilde{f}(L_1)$ and $\tilde{f}(\tilde{x})$ lies on $\tilde{f}(L')$, so we conclude that the curve $\tilde{f}(L_1)$ is at a bounded distance from $\tilde{f}(L')$. Therefore $\tilde{f}(L_1)$ limits to a point on S^1 and moreover, the forward endpoints of $\tilde{f}(L_1)$ and $\tilde{f}(L')$ are the same, namely $(f_*(\gamma))_\infty$. Thus $\tilde{f}(L)$ has forward endpoint $(f_*(\gamma))_\infty$ on the circle at infinity. □

Lemma 4.8. *Keep the hypotheses of Proposition 4.1, and consider the function $\partial\tilde{f}$ of Definition 4.5, defined on the subset $(\Gamma_X)_\infty$. Then $\partial\tilde{f}$ is monotonic, that is, it preserves the* *circular order on S^1. In particular, if $a,b,c \in (\Gamma_X)_\infty$ and (a,b,c) is a positively oriented triple, then so is the triple $(\partial\tilde{f}(a), \partial\tilde{f}(b), \partial\tilde{f}(c))$.*

Proof. Suppose $a,b,c \in (\Gamma_X)_\infty$ such that (a,b,c) is a positively oriented triple. Let L_1, L_2, L_3 be oriented geodesic lines joining a to b, b to c and c to a respectively, and let T be the ideal geodesic triangle with vertices a,b,c. Then by Lemma 4.7, $\tilde{f}$ maps T to an ideal triangle $\tilde{f}(T)$ (whose sides $\tilde{f}(L_1), \tilde{f}(L_2), \tilde{f}(L_3)$ are not necessarily geodesics) with vertices at $\partial\tilde{f}(a), \partial\tilde{f}(b), \partial\tilde{f}(c)$. Now L_1, L_2, L_3 form the boundary of the triangle T, with its induced border orientation. Since f is orientation preserving and hence so is $\tilde{f}$, we conclude that the orientation of $\tilde{f}(L_1) \cup \tilde{f}(L_2) \cup \tilde{f}(L_3)$ matches the orientation induced as the boundary of the triangle $\tilde{f}(T)$. Therefore the vertices $(\partial\tilde{f}(a), \partial\tilde{f}(b), \partial\tilde{f}(c))$ of the image triangle $\tilde{f}(T)$ form a positively oriented triple. This concludes the proof of monotonicity of $\partial\tilde{f}$. ☐

Lemma 4.9. *Keep the hypotheses of Proposition 4.1, and consider the function $\partial\tilde{f}$ of Definition 4.5, defined on the subset $(\Gamma_X)_\infty$. Then $\partial\tilde{f}$ can be extended uniquely to a homeomorphism of S^1, hence part (1) of the proposition.*

Proof. Note that $(\Gamma_X)_\infty$ and $(\Gamma_Y)_\infty$ are both dense in S^1 by Proposition 3.1, since X and Y are complete, Nielsen-convex hyperbolic surfaces. Since $\partial\tilde{f}$ is monotonic, it can be extended uniquely and continuously to the closure $\overline{(\Gamma_X)_\infty}$, unless there are jump discontinuities. However the image $\partial\tilde{f}((\Gamma_X)_\infty) = (\Gamma_Y)_\infty$ is dense in S^1, and therefore there cannot be any jump discontinuities. So $\partial\tilde{f}$ can be extended uniquely and continuously to the closure $\overline{(\Gamma_X)_\infty}$, which is S^1. Thus the extension is defined over all of S^1, and $\partial\tilde{f} \colon S^1 \to S^1$ is continuous. The function $\partial\tilde{f}$ is monotonic because $\partial\tilde{f}|_{(\Gamma_X)_\infty}$ is monotonic, and hence injective. The image $\partial\tilde{f}(S^1)$ is compact and hence closed in S^1, but contains the dense set $(\Gamma_Y)_\infty$, and so is all of S^1, making $\partial\tilde{f}$ surjective. Thus $\partial\tilde{f}$ is a continuous bijection between one dimensional manifolds, so is a homeomorphism of S^1, proving part (1) of the proposition. ☐

Lemma 4.10. *Keep the hypotheses of Proposition 4.1, and consider the function $\partial\tilde{f}$ of Lemma 4.9. Let $\hat{f}$ be the function which equals $\tilde{f}$ on $\mathbb{H}^2$ and $\partial\tilde{f}$ on S^1. Then $\hat{f}$ is a homeomorphism of $\mathbb{H}^2 \cup S^1$, that is, part (2) of the proposition.*

Proof. Note that $\hat{f}$ is already continuous on the open set $\mathbb{H}^2$ of $\mathbb{H}^2 \cup S^1$ and has a continuous inverse on its image $\mathbb{H}^2$, being equal to the homeomorphism $\tilde{f}$ there. It remains to show that $\hat{f}$ is continuous at points of S^1, so that its inverse, being equal to $\widehat{f^{-1}}$ will also be continuous at points of S^1, concluding the proof. Toward that end, let q be an arbitrary point on S^1, at which we prove the continuity of $\hat{f}$. For three distinct points $a,b,c \in S^1$, we denote by $H(a,b;c)$ the union of the open hyperbolic half plane on the c side of the geodesic joining a and b, and the open arc containing c of the circle at infinity cut out by a and b. A basis of neighbourhoods at $\hat{f}(q)$ in $\mathbb{H}^2 \cup S^1$ is the set $\{H(a,b;\hat{f}(q)) | a,b \in (\Gamma_Y)_\infty, \text{ and } a,b,\hat{f}(q) \text{ are distinct}\}$, since $(\Gamma_Y)_\infty$ is dense in S^1 by Proposition 3.1.

Let a,b be arbitrary distinct points in $(\Gamma_Y)_\infty$. We need to show that $\hat{f}^{-1}H(a,b;\hat{f}(q))$ is an open set in $\mathbb{H}^2 \cup S^1$ containing q. By Lemma 4.7 applied to $\tilde{f}^{-1}$, the $\hat{f}$-preimage of the geodesic line joining a and b (including the endpoints a and b) is a curve embedded in $\mathbb{H}^2 \cup S^1$ joining $\partial\tilde{f}^{-1}(a)$ and $\partial\tilde{f}^{-1}(b)$ whose interior lies entirely in $\mathbb{H}^2$. This curve separates $\mathbb{H}^2 \cup S^1$ into two open sets, and we denote by U the one which contains q. Since $\tilde{f}$ is a homeomorphism, it is clear that $\tilde{f}^{-1}(H(a,b;\hat{f}(q)) \cap \mathbb{H}^2) = U \cap \mathbb{H}^2$. Similarly since $\partial\tilde{f}$ is a circular order preserving homeomorphism, we have $\partial\tilde f^{-1}(H(a,b;\hat f(q))\cap S^1) = U\cap S^1$. Then it is clear that $\hat f^{-1}H(a,b;\hat f(q)) = U$, which is an open set containing q, concluding the proof of part (2) of the proposition. □

Proof of Proposition 4.1 parts (3)–(6). The function $\partial\tilde f$ is equivariant because it is a continuous extension of the equivariant map $\tilde f$ to the closure of its domain. Since $f_*(\gamma)$ ranges over all of Γ_Y as γ ranges over Γ_X, we collect the two sides of the equality $(\partial\tilde f)\circ\gamma = f_*(\gamma)\circ(\partial\tilde f)$ into an equality of sets $(\partial\tilde f)\circ\Gamma_X = \Gamma_Y\circ(\partial\tilde f)$ in $\mathrm{PSL}(2,\mathbb{R})$, proving part (3). Next, we prove part (4). If f_t is an isotopy which lifts to an isotopy $\tilde f_t$, then we have $f_{0*} = f_{1*}$, and hence for every hyperbolic $\gamma\in\Gamma_X$, $\partial\tilde f_0(\gamma_\infty) = f_{0*}(\gamma)_\infty = f_{1*}(\gamma)_\infty = \partial\tilde f_1(\gamma_\infty)$, using Definition 4.5. Therefore the continuous functions $\partial\tilde f_0$ and $\partial\tilde f_1$ agree on the dense set $(\Gamma_X)_\infty$, and hence are equal. Next, we prove part (5). We compute, using Definition 4.5, that for every hyperbolic $\gamma\in\Gamma_X$, $\partial(\tilde g\circ\tilde f)(\gamma_\infty) = ((g\circ f)_*(\gamma))_\infty = (g_*(f_*(\gamma)))_\infty = \partial\tilde g((f_*(\gamma))_\infty) = (\partial\tilde g)\circ(\partial\tilde f)(\gamma_\infty)$. Again the continuous functions $\partial(\tilde g\circ\tilde f)$ and $(\partial\tilde g)\circ(\partial\tilde f)$ agree on the dense set $(\Gamma_X)_\infty$, and hence are equal. For the final part (6), note that $\sigma\in\mathrm{PSL}(2,\mathbb{R})$, as an isometry of $\mathbb{H}^2$, also extends to a homeomorphism of S^1, which we are denoting by the same symbol σ as mentioned in Section 3.1. Therefore $\sigma\circ\tilde f$ extends to a homeomorphism at infinity, and $\partial(\sigma\circ\tilde f) = \sigma\circ(\partial\tilde f)$. □

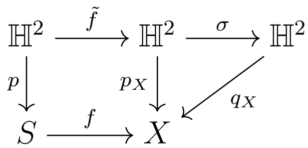
5. The Topology of the Marked Moduli Space

In this section, we define the topology on the set $\mathcal{T}(S)$ using homeomorphisms at infinity. The analogue in Teichmüller theory, the idea of representing marked Riemann surface structures by induced homeomorphisms at infinity, has existed since the early days of Teichmüller theory (see [Ahl63]), although obtaining the topology from homeomorphisms at infinity is a delicate matter. To define the topology, fix a universal cover $p\colon\mathbb{H}^2\to S$ with deck group $\Gamma\subset\mathrm{PSL}(2,\mathbb{R})$, which is a torsion-free Fuchsian group of the first kind. The universal cover p induces a homeomorphism between S and $\Gamma\backslash\mathbb{H}^2$, which is a complete Nielsen-convex hyperbolic surface by Proposition 3.1. We define a topological space $\mathcal{T}(p)$ associated to the universal cover p, and a bijective function $\Phi_p\colon\mathcal{T}(S)\to\mathcal{T}(p)$, which we then declare to be a homeomorphism, yielding a topology on $\mathcal{T}(S)$.

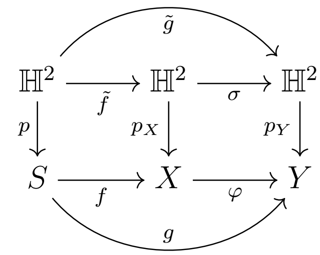
Definition 5.1 (The space $\mathcal{T}(p)$)**.**

$$\begin{aligned}\mathcal{T}(p) &\coloneqq \mathrm{PSL}(2,\mathbb{R})\Big\backslash\widetilde{\mathcal{T}}(p) \quad \textit{where}\\ \widetilde{\mathcal{T}}(p) &\coloneqq \left\{F\in\mathrm{Homeo}^+(S^1)\,\middle|\,F\circ\Gamma\circ F^{-1}\subset\mathrm{PSL}(2,\mathbb{R})\right\}\end{aligned}\tag{5}$$

$\mathcal{T}(p)$ is naturally a Hausdorff topological space as follows. $\widetilde{\mathcal{T}}(p)$ inherits a topology as a subspace of $\mathrm{Homeo}^+(S^1)$ with the compact-open topology. $\mathcal{T}(p)$ has the quotient topology, where $\mathrm{PSL}(2,\mathbb{R})$ acts on $\widetilde{\mathcal{T}}(p)$ by composition on the left. Since $\mathrm{Homeo}^+(S^1)$ is a topological group and $\mathrm{PSL}(2,\mathbb{R})$ is a closed subgroup, the right coset space $\mathrm{Homeo}^+(S^1)\Big\backslash\mathrm{PSL}(2,\mathbb{R})$ is Hausdorff (see [Mun00, p146, 7(d)]). Therefore its subspace $\mathcal{T}(p)$ is also Hausdorff. We denote by π_p the quotient map $\widetilde{\mathcal{T}}(p)\to\mathcal{T}(p)$.



a. $\Phi_p[X, f]$ is independent of the cover $p_X : \mathbb{H}^2 \to X$.



b. $\Phi_p[X, f]$ is independent of the representative (X, f) of the marked hyperbolic structure.

FIGURE 4. $\Phi_p[X, f]$ is well defined.

Definition 5.2 (The function Φ_p)**.** *Suppose $[X, f] \in \mathcal{T}(S)$ is a marked hyperbolic structure. Choose a universal cover $p_X\colon \mathbb{H}^2 \to X$. The marking map $f\colon S \to X$ lifts to a homeomorphism $\tilde{f}\colon \mathbb{H}^2 \to \mathbb{H}^2$, which, by Proposition 4.1(1), extends to a homeomorphism at infinity $\partial\tilde{f}\colon S^1 \to S^1$. We define $\Phi_p[X, f]$ to be the right coset $[\partial\tilde{f}] = \mathrm{PSL}(2,\mathbb{R}) \circ (\partial\tilde{f})$ of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{Homeo}^+(S^1)$.*

Lemma 5.3. *For each $[X, f] \in \mathcal{T}(S)$, $\Phi_p[X, f]$ is a well defined element of $\mathcal{T}(p)$. Thus the definition above yields a function $\Phi_p\colon \mathcal{T}(S) \to \mathcal{T}(p)$.*

Proof. First we need to show that $\Phi_p[X, f]$ is independent of the three choices made in the definition of Φ_p, namely, the choice of the representative (X, f) of the marked hyperbolic structure, the choice of the universal cover p_X, and the choice of the lift $\tilde{f}$. We treat these one at a time, and in reverse order.

(1) The choice of the lift $\tilde{f}$ of f: Any other lift of f is of the form $\sigma \circ \tilde{f}$, where $\sigma \in \Gamma_X$ is a deck transformation. This extends to the homeomorphism at infinity $\sigma \circ (\partial\tilde{f})$, by Proposition 4.1(6). Since $\sigma \in \mathrm{PSL}(2,\mathbb{R})$, we have an equality of right cosets $\mathrm{PSL}(2,\mathbb{R}) \circ (\sigma \circ (\partial\tilde{f})) = \mathrm{PSL}(2,\mathbb{R}) \circ (\partial\tilde{f})$. That is, $[\partial(\sigma \circ \tilde{f})] = [\sigma \circ (\partial\tilde{f})] = [\partial\tilde{f}]$.

(2) The choice of the universal cover $p_X\colon \mathbb{H}^2 \to X$: Suppose that $q_X\colon \mathbb{H}^2 \to X$ is another universal cover. Since universal covers are unique up to isometry, there is a $\sigma \in \mathrm{PSL}(2,\mathbb{R})$ such that $p_X = q_X \circ \sigma$. See the diagram in Figure 4a, which commutes. Then $\sigma \circ \tilde{f}$ is a lift of f with respect to the universal covers p and q_X, which extends to the homeomorphism at infinity $\sigma \circ (\partial\tilde{f})$, by Proposition 4.1(6). Again we have $[\partial(\sigma \circ \tilde{f})] = [\partial\tilde{f}]$.

(3) The choice of the representative (X, f) of the marked hyperbolic structure: Suppose (Y, g) is another representative of the same marked hyperbolic structure. Then there is an isometry $\varphi\colon X \to Y$ such that $\varphi \circ f$ is homotopic to g. Then φ lifts to an isometry $\sigma \in \mathrm{PSL}(2,\mathbb{R})$. For a lift $\tilde{f}$ of f, $\sigma \circ \tilde{f}$ is a lift of $\varphi \circ f$. The homotopy from $\varphi \circ f$ to g lifts to a homotopy from $\sigma \circ \tilde{f}$ to a lift $\tilde{g}$ of g. See the diagram in Figure 4b, in which the squares commute and the top and bottom triangles homotopy commute. By Proposition 4.1, parts (4) and (6), we have $\partial\tilde{g} = \partial(\sigma \circ \tilde{f}) = \sigma \circ (\partial\tilde{f})$. Therefore again we have $[\partial\tilde{g}] = [\sigma \circ (\partial\tilde{f})] = [\partial\tilde{f}]$, so that $\Phi_p[X, f]$ is a well defined right coset of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{Homeo}^+(S^1)$.

Next, to show that $\Phi_p[X,f]$ is actually an element of the subspace $\mathcal{T}(p)$ of the right coset space $\mathrm{PSL}(2,\mathbb{R})\backslash\mathrm{Homeo}^+(S^1)$, we use Proposition 4.1(3) to obtain $(\partial\tilde{f})\circ\Gamma = \Gamma_X\circ(\partial\tilde{f})$, or in other words, $(\partial\tilde{f})\circ\Gamma\circ(\partial\tilde{f})^{-1} = \Gamma_X \subset \mathrm{PSL}(2,\mathbb{R})$. Therefore $\partial\tilde{f}\in\widetilde{\mathcal{T}}(p)$, and $[\partial\tilde{f}]\in\mathcal{T}(p)$. So the codomain of Φ_p is $\mathcal{T}(p)$ indeed. □

Lemma 5.4. *$\Phi_p\colon \mathcal{T}(S)\to\mathcal{T}(p)$ is a bijection.*

Proof. To see that Φ_p is injective, suppose $\Phi_p[X,f]=\Phi_p[Y,g]$. Let $\tilde{f},\tilde{g}$ denote the lifts of f,g with respect to universal covers $p_X\colon\mathbb{H}^2\to X$ and $p_Y\colon\mathbb{H}^2\to Y$. Note that $(\partial\tilde{f})\circ\Gamma\circ(\partial\tilde{f})^{-1} = \Gamma_X$ and $(\partial\tilde{g})\circ\Gamma\circ(\partial\tilde{g})^{-1}=\Gamma_Y$ by Proposition 4.1(3). Also, $[\partial\tilde{g}]=[\partial\tilde{f}]$, so there exists an element σ of $\mathrm{PSL}(2,\mathbb{R})$ such that $\partial\tilde{g}=\sigma\circ(\partial\tilde{f})$. Thus we have $\sigma\circ\Gamma_X=\sigma\circ(\partial\tilde{f})\circ\Gamma\circ(\partial\tilde{f})^{-1} = (\partial\tilde{g})\circ\Gamma\circ(\partial\tilde{f})^{-1}=\Gamma_Y\circ(\partial\tilde{g})\circ(\partial\tilde{f})^{-1}=\Gamma_Y\circ\sigma$. Thus σ is equivariant, not only as a homeomorphism of S^1, but also as an isometry of $\mathbb{H}^2$, and hence descends to an isometry $\varphi\colon X\to Y$. Further, for every $\gamma\in\Gamma$, we have $(\varphi\circ f)_*(\gamma)=(\sigma\circ(\partial\tilde{f}))\circ\gamma\circ(\sigma\circ(\partial\tilde{f}))^{-1} = (\partial\tilde{g})\circ\gamma\circ(\partial\tilde{g})^{-1}=g_*(\gamma)$. Therefore $(\varphi\circ f)_*$ and g_* are equal as isomorphisms from Γ to Γ_Y, and hence are equal as the π_1 functor as well. Since S is a $K(\pi_1(S),1)$ classifying space, $\varphi\circ f$ is homotopic to g. Therefore $[X,f]=[Y,g]$ in $\mathcal{T}(S)$, so Φ_p is injective.

To see that Φ_p is surjective, suppose $[F]\in\mathcal{T}(p)$. Define $\Gamma_X=F\circ\Gamma\circ F^{-1}$. Since $F\in\widetilde{\mathcal{T}}(p)$, we have $\Gamma_X\subset\mathrm{PSL}(2,\mathbb{R})$. Γ_X is discrete because conjugation by F is a homeomorphism of $\mathrm{Homeo}^+(S^1)$, and because Γ is also discrete. Γ_X is also torsion-free because it is isomorphic to Γ, which is torsion-free. Further, F maps the limit set of Γ to the limit set of Γ_X, which is therefore all of S^1. Hence $X=\Gamma_X\backslash\mathbb{H}^2$ is a complete, Nielsen-convex hyperbolic surface by Proposition 3.1. The Douady–Earle extension (recall Theorem 3.5) of F is conformally natural and thus equivariant, hence descends to a homeomorphism $f\colon S\to X$. Clearly $\Phi_p[X,f]=[F]$, so Φ_p is surjective. This completes the proof of bijectivity of Φ_p. □

The above proposition allows us to define a topology on $\mathcal{T}(S)$, and then we check that it actually does not depend on the choice of the universal cover p that we used in the definition.

Definition 5.5 (Topology of the Marked Moduli Space)**.** *The topology on $\mathcal{T}(S)$ is defined by declaring the bijection $\Phi_p\colon\mathcal{T}(S)\to\mathcal{T}(p)$ to be a homeomorphism. In other words, a subset $U\subset\mathcal{T}(S)$ is open if and only if $\Phi_p(U)\subset\mathcal{T}(p)$ is open.*

Proposition 5.6. *If $p'\colon\mathbb{H}^2\to S$ is another universal cover with deck group $\Gamma'\subset\mathrm{PSL}(2,\mathbb{R})$, a torsion-free Fuchsian group of the first kind, then $\Phi_{p'}\circ\Phi_p^{-1}\colon\mathcal{T}(p)\to\mathcal{T}(p')$ is a homeomorphism. Thus the topology on $\mathcal{T}(S)$ does not depend on the choice of the universal cover p. Moreover, $\mathcal{T}(S)$ is a Hausdorff space.*

Proof. The identity map of S lifts, with respect to the covers p' and p, to a map $h\colon\mathbb{H}^2\to\mathbb{H}^2$ such that $p\circ h=p'$. By Proposition 4.1(1), h extends to a homeomorphism at infinity ∂h. Note that h is not necessarily an isometry, because the two universal covers p and p' can possibly induce different hyperbolic metrics on S, although both are complete and Nielsen-convex by Proposition 3.1.

Let $[X,f]\in\mathcal{T}(S)$ be a marked hyperbolic structure and $p_X\colon\mathbb{H}^2\to X$ be a universal cover. The marking map f lifts to $\tilde{f}$ with respect to the covers p and p_X, so that $\Phi_p[X,f]=[\partial\tilde{f}]$. That is, the right $\mathrm{PSL}(2,\mathbb{R})$-coset $\Phi_p[X,f]$ is represented by the homeomorphism $\partial\tilde{f}$. Then the map $\tilde{f}'$ defined by $\tilde{f}'=\tilde{f}\circ h$ is a lift of f with respect to the covers p' and p_X. See

a. The identity lifts to the map h with respect to covers p' and p.

b. $\Phi_{p'} \circ \Phi_p^{-1}$ is induced by multiplication by ∂h on the right.

FIGURE 5. $\Phi_{p'} \circ \Phi_p^{-1}$ is a homeomorphism.

the diagram in Figure 5a, which commutes. Thus $\Phi_{p'}[X, f]$ is represented by $\partial(\tilde{f} \circ h)$, which equals $(\partial \tilde{f}) \circ (\partial h)$ due to Proposition 4.1(5). Thus $\Phi_{p'} \circ \Phi_p[\partial \tilde{f}] = [\partial \tilde{f}] \circ (\partial h)$. In other words, $\Phi_{p'} \circ \Phi_p^{-1} \colon \mathcal{T}(p) \to \mathcal{T}(p')$ is induced simply by multiplication on the right by ∂h. Multiplication on the right by the fixed map ∂h is a homeomorphism of the topological group $\mathrm{Homeo}^+(S^1)$, and it restricts to a homeomorphism from $\widetilde{\mathcal{T}}(p)$ to $\widetilde{\mathcal{T}}(p')$, which descends to a homeomorphism $\Phi_{p'} \circ \Phi_p^{-1}$ of the spaces of right cosets $\mathcal{T}(p)$ and $\mathcal{T}(p')$. See the diagram in Figure 5b, which commutes, and in which the top two vertical arrows are embeddings and the bottom two vertical arrows are quotient maps.

Now we can show that the topology of $\mathcal{T}(S)$ does not depend on the universal cover p. Indeed, for another universal cover p' and any subset $U \subset \mathcal{T}(S)$, the condition $\Phi_p(U)$ is open in $\mathcal{T}(p)$ is equivalent, since $\Phi_{p'} \circ \Phi_p^{-1}$ is a homeomorphism, to the condition that $\Phi_{p'}(U) = \Phi_{p'} \circ \Phi_p^{-1}(\Phi_p(U))$ is open in $\mathcal{T}(p')$.

Finally, $\mathcal{T}(S)$ is Hausdorff because it is homeomorphic to $\mathcal{T}(p)$, which is Hausdorff as remarked after Definition 5.1. □

6. Continuity of the Action

Recall the action function A, and the action $A_{[\psi]}$ of a mapping class $[\psi]$, from equation (2). Now that we have a well defined topology on $\mathcal{T}(S)$, we can ask if the action of $\mathrm{MCG}(S)$ on $\mathcal{T}(S)$ is continuous, and in this section, we answer it in the affirmative. We can also ask if the action representation $\mathrm{MCG}(S) \to \mathrm{Homeo}(\mathcal{T}(S))$ is continuous. However, we have not defined any topology on the codomain. For infinite type surfaces S, we do not expect $\mathcal{T}(S)$ to be locally compact in general. So the compact-open topology may not be very useful. Instead we prove that the action function $A \colon \mathrm{MCG}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$ is continuous. It will follow that for each mapping class $[\psi] \in \mathrm{MCG}(S)$, its action on $\mathcal{T}(S)$, that is, the function $A_{[\psi]} \colon \mathcal{T}(S) \to \mathcal{T}(S)$, is a homeomorphism. We remark that the continuity of the action function A is straightforward for the mapping class group of a finite type surface acting on the Teichmüller space. In this case, the mapping class group is discrete, and so A is continuous if and only if for all $[\psi] \in \mathrm{MCG}(S)$, $A_{[\psi]}$ is continuous. But this is true because $A_{[\psi]}$ is an isometry. In order to prove continuity of A, we must compute A in terms of homeomorphisms at infinity, since the topology on $\mathcal{T}(S)$ is defined via homeomorphisms at infinity. This is achieved by showing that mapping classes induce homeomorphisms at infinity as well. The idea of studying homeomorphisms of surfaces via the induced homeomorphisms at infinity dates to work of Nielsen (see [HT85] and the references therein).

First we fix a universal cover $p\colon \mathbb{H}^2 \to S$ with deck group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$, a torsion-free Fuchsian group of the first kind as before. Recall that Γ is also naturally a subgroup of $\mathrm{Homeo}^+(S^1)$. Let $\mathrm{N}(\Gamma)$ be the normaliser of Γ in $\mathrm{Homeo}^+(S^1)$ and let π_N be the quotient map $\mathrm{N}(\Gamma) \to \mathrm{N}(\Gamma)/\Gamma$. We now define a function $\Psi_p\colon\ \mathrm{MCG}(S) \to \mathrm{N}(\Gamma)/\Gamma$ which will enable us to compute the action function in terms of homeomorphisms at infinity.

Definition 6.1 (The function Ψ_p)**.** *Suppose $[\psi] \in \mathrm{MCG}(S)$ is a mapping class. The homeomorphism ψ lifts to $\tilde{\psi}\colon \mathbb{H}^2 \to \mathbb{H}^2$ which, by Proposition 4.1(1), extends to a homeomorphism at infinity $\partial\tilde{\psi}\colon S^1 \to S^1$. We define $\Psi_p[\psi]$ to be the right coset $[\partial\tilde{\psi}] = \Gamma \circ (\partial\tilde{\psi})$ of Γ in* $\mathrm{Homeo}^+(S^1)$.

Lemma 6.2. *For each $[\psi] \in \mathrm{MCG}(S)$, $\Psi_p[\psi]$ is a well defined element of* $\mathrm{N}(\Gamma)/\Gamma$. *Thus the definition above yields a function $\Psi_p\colon\ \mathrm{MCG}(S) \to \mathrm{N}(\Gamma)/\Gamma$.*

Proof. First, we need to show that $\Psi_p[\psi]$ is independent of two choices made in the definition of Ψ_p, namely, the choice of the representative homeomorphism ψ of the mapping class, and the choice of its lift $\tilde{\psi}$. We treat these one at a time, and in reverse order.

(1) The choice of the lift $\tilde{\psi}$ of ψ: Any other lift of ψ is of the form $\sigma \circ \tilde{\psi}$, where $\sigma \in \Gamma$ is a deck transformation. This extends to the homeomorphism at infinity $\sigma \circ (\partial\tilde{\psi})$, by Proposition 4.1(6). Since $\sigma \in \Gamma$, we have an equality of right cosets $\Gamma \circ (\sigma \circ (\partial\tilde{\psi})) = \Gamma \circ (\partial\tilde{\psi})$. That is, $[\partial(\sigma \circ \tilde{\psi})] = [\sigma \circ (\partial\tilde{\psi})] = [\partial\tilde{\psi}]$.

(2) The choice of the representative ψ of the mapping class $[\psi]$: Suppose ψ' is another representative of the same mapping class. Then ψ is homotopic to ψ'. This homotopy lifts to a homotopy between $\tilde{\psi}$ and a lift $\tilde{\psi}'$ of ψ'. By Proposition 4.1(4), we have $\partial\tilde{\psi}' = \partial\tilde{\psi}$, so $[\partial\tilde{\psi}] = [\partial\tilde{\psi}]$, so that $\Psi_p[\psi]$ is a well defined right coset of Γ in $\mathrm{Homeo}^+(S^1)$.

Next, to show that $\Psi_p[\psi]$ is actually an element of the subspace $\mathrm{N}(\Gamma)/\Gamma$ of the right coset space $\Gamma\backslash\mathrm{Homeo}^+(S^1)$, we use Proposition 4.1(3) to obtain $\Gamma \circ (\partial\tilde{\psi}) = (\partial\tilde{\psi}) \circ \Gamma$, or in other words, $(\partial\tilde{\psi}) \circ \Gamma \circ (\partial\tilde{\psi})^{-1} = \Gamma$. Therefore $\partial\tilde{\psi}$ lies in the normaliser $\mathrm{N}(\Gamma)$, and $[\partial\tilde{\psi}] \in \mathrm{N}(\Gamma)/\Gamma$. So the codomain of Ψ_p is $\mathrm{N}(\Gamma)/\Gamma$ indeed. ☐

Note that the left and right cosets of Γ by $\partial\tilde{\psi}$ are equal, and our notation $\mathrm{N}(\Gamma)/\Gamma$ suggests left coset space. This is in keeping with the convention for group quotients by normal subgroups.

Lemma 6.3. $\Psi_p\colon\ \mathrm{MCG}(S) \to \mathrm{N}(\Gamma)/\Gamma$ *is a group isomorphism.*

Proof. A similar argument as with Φ_p in Lemma 5.4 shows that Ψ_p is a bijection. To see that Ψ_p is a group homomorphism, suppose that $\psi_1, \psi_2 \in \mathrm{Homeo}^+(S)$ are homeomorphisms with lifts $\tilde{\psi}_1, \tilde{\psi}_2$ to the universal cover. Then $\tilde{\psi}_1 \circ \tilde{\psi}_2$ is a lift of $\psi_1 \circ \psi_2$, and we compute $\Psi_p([\psi_1] \cdot [\psi_2]) = \Psi_p[\psi_1 \circ \psi_2] = [\partial(\tilde{\psi}_1 \circ \tilde{\psi}_2)]$ which equals $[(\partial\tilde{\psi}_1) \circ (\partial\tilde{\psi}_2)]$ by Proposition 4.1(5), and hence equals $[\partial\tilde{\psi}_1] \circ [\partial\tilde{\psi}_2] = \Psi_p[\psi_1] \cdot \Psi_p[\psi_2]$. Hence Ψ_p is a group homomorphism and, since it is a bijection as well, is a group isomorphism. ☐

We are ready to compute the action function in terms of homeomorphisms at infinity. The general idea is that $\Phi_p(A([\psi], [X, f])) = \Phi_p[X, f] \circ (\Psi_p[\psi])^{-1}$. However, $\Phi_p[X, f]$ and $\Psi_p[\psi]$ are right cosets of $\mathrm{PSL}(2,\mathbb{R})$ and Γ respectively in $\mathrm{Homeo}^+(S^1)$ and $\mathrm{N}(\Gamma)$ respectively. In order to avoid dealing with multiplication of cosets, we state the computation in the following manner:

Proposition 6.4. *Suppose* $[X, f] \in \mathcal{T}(S)$ *and* $[\psi] \in \mathrm{MCG}(S)$. *If* $\Phi_p[X, f] = [F]$ *and* $\Psi_p[\psi] = [G]$, *then we have*

$$\Phi_p(A([\psi], [X, f])) = [F \circ G^{-1}] \tag{6}$$

Proof. Let $\tilde{f}$ be a lift of the marking map f, and let $\tilde{\psi}$ be a lift of the homeomorphism ψ of S. Then $\Phi_p[X, f] = [\partial\tilde{f}]$ and $\Psi_p[\psi] = [\partial\tilde{\psi}]$. We have $A([\psi], [X, f]) = [X, f \circ \psi^{-1}]$. Then the modified marking map $f \circ \psi^{-1}$ lifts to the map $\tilde{f} \circ \tilde{\psi}^{-1}$ which, by Proposition 4.1(5), extends to the homeomorphism at infinity $\partial(\tilde{f} \circ \tilde{\psi}^{-1}) = (\partial\tilde{f}) \circ (\partial\tilde{\psi})^{-1}$. Therefore $\Phi_p(A([\psi], [X, f])) = \Phi_p[X, f \circ \psi^{-1}] = [(\partial\tilde{f}) \circ (\partial\tilde{\psi})^{-1}]$. If F is any other representative of the right coset $[\partial\tilde{f}]$ of $\mathrm{PSL}(2, \mathbb{R})$ in $\mathrm{Homeo}^+(S^1)$, then $F = \sigma \circ (\partial\tilde{f})$ for some $\sigma \in \mathrm{PSL}(2, \mathbb{R})$. Similarly if G is any other representative of the (left or right) coset $[\partial\tilde{\psi}]$ of Γ in $\mathrm{N}(\Gamma)$, then $G = (\partial\tilde{\psi}) \circ \gamma^{-1}$ for some $\gamma \in \Gamma$. Then $F \circ G^{-1} = (\sigma \circ (\partial\tilde{f})) \circ ((\partial\tilde{\psi}) \circ \gamma^{-1})^{-1} = \sigma \circ (\partial\tilde{f}) \circ \gamma \circ (\partial\tilde{\psi})^{-1}$, which equals, by Proposition 4.1(3), $\sigma \circ f_*(\gamma) \circ (\partial\tilde{f}) \circ (\partial\tilde{\psi})^{-1}$. Since $\sigma, f_*(\gamma) \in \mathrm{PSL}(2, \mathbb{R})$, we have $[F \circ G^{-1}] = [(\partial\tilde{f}) \circ (\partial\tilde{\psi})^{-1}]$ as right cosets of $\mathrm{PSL}(2, \mathbb{R})$ in $\mathrm{Homeo}^+(S^1)$. In other words, $\Phi_p(A([\psi], [X, f])) = [F \circ G^{-1}]$. □

Thus in terms of homeomorphisms at infinity, the action of the mapping class group is simply by multiplication on the right by the inverse. Since inversion and multiplication are continuous operations in the topological group $\mathrm{Homeo}^+(S^1)$, the formula $(G, F) \mapsto F \circ G^{-1}$ defines a continuous function on $\mathrm{N}(\Gamma) \times \widetilde{\mathcal{T}}(p) \subset \mathrm{Homeo}^+(S^1) \times \mathrm{Homeo}^+(S^1)$. This descends to the action function A, which is continuous as long as $[\partial\tilde{\psi}]$ depends continuously on $[\psi]$. That is, we need to show that Ψ_p is continuous, which we now do in Lemma 6.5.

Lemma 6.5. $\Psi_p \colon \mathrm{MCG}(S) \to \mathrm{N}(\Gamma)/\Gamma$ *is a homeomorphism.*

Proof. Fix a basepoint $\tilde{s} \in \mathbb{H}^2$ and let $s = p(\tilde{s})$. Let d be any path metric on S^1, for example, the visual metric on S^1 induced by the basepoint $\tilde{s}$. Since we have fixed a universal cover $p \colon \mathbb{H}^2 \to S$, we have access to the complete hyperbolic metric d_S on S and its injectivity radius function inj_S, which is a continuous function on S.

For the continuity of Ψ_p, we only need to show that Ψ_p is continuous at the identity $[\mathrm{id}_S] \in \mathrm{MCG}(S)$, since Ψ_p is a group isomorphism. Note that $\Psi_p[\mathrm{id}_S] = [\partial \mathrm{id}_{\mathbb{H}^2}] = [\mathrm{id}_{S^1}]$. Let U be an open neighbourhood of $[\mathrm{id}_{S^1}]$ in $\mathrm{N}(\Gamma)/\Gamma$. We have to show that $\Psi_p^{-1}(U)$ is an open neighbourhood of $[\mathrm{id}_S]$ in $\mathrm{MCG}(S)$. Since the quotient $\pi_N \colon \mathrm{N}(\Gamma) \to \mathrm{N}(\Gamma)/\Gamma$ is continuous, the preimage $\pi_N^{-1}(U)$ is open in $\mathrm{N}(\Gamma)$, and hence is the intersection with $\mathrm{N}(\Gamma)$ of an open neighbourhood U_1 of id_{S^1} in $\mathrm{Homeo}^+(S^1)$. Therefore there is an ε such that the ball in $\mathrm{Homeo}^+(S^1)$ (in the metric of uniform convergence)

$$B_d(\mathrm{id}_{S^1}, \varepsilon) = \{\theta \in \mathrm{Homeo}^+(S^1) \mid d(\theta(q), q) < \varepsilon \text{ for all } q \in S^1\}$$

is contained in U_1. Reducing ε if necessary, assume that ε is less than half the d-length of S^1.

We will construct an open neighbourhood V_1 of id_S in $\mathrm{Homeo}^+(S)$ and for all $\psi \in V_1$, construct a lift $\tilde{\psi}$ such that $\partial\tilde{\psi} \in B_d(\mathrm{id}_{S^1}, \varepsilon)$. Since $B_d(\mathrm{id}_{S^1}, \varepsilon) \subset U_1$, this means that $\Psi_p[\psi] = [\partial\tilde{\psi}] \in \pi_N(U_1 \cap \mathrm{N}(\Gamma)) = U$. Since quotienting by the group action of $\mathrm{Homeo}_0(S)$ is an open map, the projection V of V_1 in $\mathrm{MCG}(S)$ is an open set. It follows that V is an open neighbourhood of $[\mathrm{id}_S]$ and is contained in $\Psi_p^{-1}(U)$. It will follow that Ψ_p is continuous at $[\mathrm{id}_S]$.

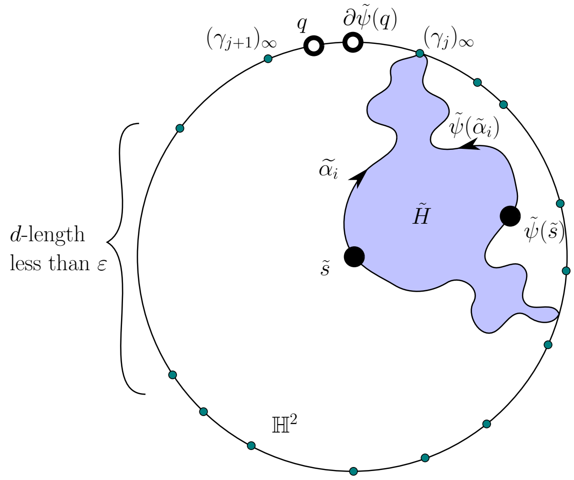


FIGURE 6. Lifts of the homeomorphisms in V_1 and their extensions to the circle at infinity.

To obtain the open neighbourhood V_1, we choose a finite set of points in $(\Gamma)_\infty$ which is $\frac{\varepsilon}{2}$-dense in S^1. This is possible because S^1 is compact and $(\Gamma)_\infty$ is dense in S^1 due to Proposition 3.1. Represent these points as sinks $(\gamma_1)_\infty, (\gamma_2)_\infty, \ldots, (\gamma_n)_\infty$ of hyperbolic elements $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$. Relabelling if necessary, and removing duplicates, assume that $(\gamma_1)_\infty, (\gamma_2)_\infty, \ldots (\gamma_n)_\infty$ are distinct and in positive circular order. These points divide the circle at infinity into n intervals, each of length at most ε. For each $j = 1, 2, \ldots, n$, choose an oriented closed curve α_j in S based at the point s such that the holonomy around α_j is $\varphi[\alpha_j] = \gamma_j$, where $\varphi\colon \pi_1(S, s) \to \Gamma$ is the holonomy representation induced by the pointed universal cover $p\colon (\mathbb{H}^2, \tilde{s}) \to (S, s)$. Since the set $\bigcup_{j=1}^n \alpha_j$ is a compact set, the continuous function inj_S has a minimum value $\delta > 0$ on it. We define V_1 to be the set

$$V_1 = \left\{ \psi \in \mathrm{Homeo}^+(S) \,\middle|\, d_S(\psi(q), q) < \delta \text{ for all } q \in \bigcup_{j=1}^{n} \alpha_j \right\}$$

which is a basic open neighbourhood of id_S of $\mathrm{Homeo}^+(S)$ in the topology of uniform convergence on compact sets.

Now let $\psi \in V_1$. Since $d_S(\psi(s), s) < \delta \le \mathrm{inj}_S(s)$, there is exactly one lift of $\psi(s)$ in the δ-neighbourhood of the lift $\tilde{s}$ of s. Define $\tilde{\psi}$ to be the unique lift of ψ that maps $\tilde{s}$ to this point. Further, for each $j = 1, 2, \ldots, n$ and each $q \in \alpha_j$, since $d_S(\psi(q), q) < \delta \le \mathrm{inj}_S(q)$, there is a unique shortest geodesic segment from q to $\psi(q)$. Moving every point q along this geodesic segment at constant speed yields the straight line homotopy H from the curve α_j to the curve $\psi(\alpha_j)$ (recall Proposition 3.3). Therefore the holonomy around $\psi(\alpha_j)$ equals the holonomy γ_j around α_j, but it also equals $\psi_*(\gamma_j)$, using facts from Section 3.3. We infer that $\psi_*(\gamma_j) = \gamma_j$. Consequently, $\partial\tilde{\psi}((\gamma_j)_\infty) = (\psi_*(\gamma_j))_\infty = (\gamma_j)_\infty$, using Definition 4.5. Since j was arbitrary, this is true for all $j = 1, 2, \ldots, n$. See Figure 6, which shows the lift $\tilde{H}$ of the homotopy H. Here $\tilde{H}$ is a homotopy from the (bi-infinite) lift $\tilde{\alpha}_j$ of α_j through $\tilde{s}$ to the lift $\tilde{\psi}(\tilde{\alpha}_j)$ of $\psi(\alpha_j)$ through $\tilde{\psi}(\tilde{s})$. Note that the forward endpoints of both α_j and $\tilde{\psi}(\tilde{\alpha}_j)$ are equal, and hence $\partial\tilde{\psi}$ fixes $(\gamma_j)_\infty$.

Next, we show the membership $\partial\tilde{\psi} \in B_d(\mathrm{id}_{S^1}, \varepsilon)$ as promised. For any $q \in S^1$, if $q = (\gamma_j)_\infty$ for some j, then trivially $d(\partial\tilde{\psi}(q), q) = 0 < \varepsilon$. Otherwise, q lies in an interval component of $S^1 \setminus \{(\gamma_1)_\infty, (\gamma_2)_\infty, \ldots, (\gamma_n)_\infty\}$ bounded by two sinks $(\gamma_j)_\infty$ and $(\gamma_{j+1})_\infty$ (subscripts of the letter γ being modulo n). Since the homeomorphism $\partial\tilde{\psi}$ preserves the circular order on S^1 and fixes both $(\gamma_j)_\infty$ and $(\gamma_{j+1})_\infty$, we conclude that $\partial\tilde{\psi}(q)$ lies in the same interval as does q. Hence $d(\partial\tilde{\psi}(q), q) < \varepsilon$, as both lie in an interval of length less than ε. Thus $d(\partial\tilde{\psi}(q), \mathrm{id}_{S^1}(q)) < \varepsilon$ for all $q \in S^1$, and so $\partial\tilde{\psi} \in B_d(\mathrm{id}_{S^1}, \varepsilon)$. This concludes the proof of continuity of Ψ_p.

We show that the inverse $\Psi_p^{-1}\colon \mathrm{N}(\Gamma)/\Gamma \to \mathrm{MCG}(S)$ is continuous by describing it explicitly. Let $\mathrm{DE}\colon \mathrm{Homeo}^+(S^1) \to \mathrm{Homeo}^+(\mathbb{H}^2)$ denote the Douady-Earle extension restricted to $\mathbb{H}^2$ (recall Theorem 3.5). Suppose $G \in \mathrm{N}(\Gamma) \subset \mathrm{Homeo}^+(S^1)$ and $\gamma \in \Gamma \subset \mathrm{PSL}(2, \mathbb{R})$. Then since G normalises Γ and DE is conformally natural, we have $\mathrm{DE}(G) \circ \gamma = \mathrm{DE}(G \circ \gamma) = \mathrm{DE}(\gamma' \circ G) = \gamma' \circ \mathrm{DE}(G)$ for some $\gamma' \in \Gamma$. Note that in this chain of equalities, the γ and γ' within parentheses are homeomorphisms of S^1, those without are isometries of $\mathbb{H}^2$. Thus $\mathrm{DE}(G)$ normalises Γ, now viewed as a group of homeomorphisms of $\mathbb{H}^2$. We denote the normaliser of Γ in $\mathrm{Homeo}^+(\mathbb{H}^2)$ by $\mathrm{Homeo}^+_\Gamma(\mathbb{H}^2)$. This is precisely the set of equivariant homeomorphisms of $\mathbb{H}^2$, or equivalently, the set of lifts of homeomorphisms of S. Let $\pi\colon \mathrm{Homeo}^+_\Gamma(\mathbb{H}^2) \to \mathrm{Homeo}^+(S)$ be the function defined by $\pi(\tilde{f}) = f$ for every homeomorphism f of S and every lift $\tilde{f}$ of f, so that $\pi \circ \mathrm{DE}(G)$ is a homeomorphism of S for each $G \in \mathrm{N}(\Gamma)$. Now each $\gamma \in \Gamma$ is a deck transformation of the universal cover $p\colon \mathbb{H}^2 \to S$. Hence for each $G \in \mathrm{N}(\Gamma)$ and $\gamma \in \Gamma$, we have $\pi \circ \mathrm{DE}(G \circ \gamma) = \pi(\mathrm{DE}(G) \circ \gamma) = \pi \circ \mathrm{DE}(G)$, and so $\pi \circ \mathrm{DE}$ descends to a function $\overline{\pi\mathrm{DE}}\colon \mathrm{N}(\Gamma)/\Gamma \to \mathrm{MCG}(S)$. For $[G] \in \mathrm{N}(\Gamma)/\Gamma$, a lift of $\overline{\pi\mathrm{DE}}[G]$ to $\mathbb{H}^2$ is simply $\mathrm{DE}(G)$, which extends to the homeomorphism at infinity G, so $\Psi_p \circ \overline{\pi\mathrm{DE}}[G] = [G]$. As $[G] \in \mathrm{N}(\Gamma)/\Gamma$ was arbitrary, $\Psi_p \circ \overline{\pi\mathrm{DE}}$ is the identity on $\mathrm{N}(\Gamma)/\Gamma$. Hence $\overline{\pi\mathrm{DE}}$ is a right inverse to Ψ_p. But we already know that Ψ_p is a bijection, so in fact $\overline{\pi\mathrm{DE}} = \Psi_p^{-1}$.

We show that π is continuous. Suppose $G_i \to G$ as $i \to \infty$ in $\mathrm{Homeo}^+_\Gamma(\mathbb{H}^2)$, which has the topology of uniform convergence on compact sets. To show the $\pi(G_i) \to \pi(G)$ as $i \to \infty$ in $\mathrm{Homeo}^+(S)$ in the topology of uniform convergence on compact sets, we show the uniform convergence on an arbitrary compact set $K \subset S$. There exists a large enough closed disk $D \subset \mathbb{H}^2$ such that K is contained in $p(D)$. Since D is compact, $G_i|_D \to G|_D$ uniformly as $i \to \infty$. Now p is a Riemannian local isometry, and therefore contracts distances. Therefore $\pi(G_i)|_K \to \pi(G)|_K$ uniformly as $i \to \infty$, and so π is continuous. Since DE is also a continuous function, the composition $\pi \circ \mathrm{DE}$ is continuous, and passing the quotient, $\overline{\pi\mathrm{DE}}$ is continuous. Therefore Ψ_p^{-1} is continuous, and Ψ_p is a homeomorphism. □

We are ready to prove the continuity of the action function A. The thrust of the argument is already proved in Lemma 6.5, we only need to finish up the point set topological details.

Theorem 6.6. *The action function* $A\colon \mathrm{MCG}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$ *given by equation* (2) *is continuous.*

Proof. As before, fix a universal cover $p\colon \mathbb{H}^2 \to S$ with deck group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$, a torsion-free Fuchsian group of the first kind. See the diagram in Figure 7, which commutes. The proposition is that the function in the bottom row is continuous. This is equivalent to the continuity of the function in the third row just above it, because Φ_p is a homeomorphism by definition, and Ψ_p is a homeomorphism by Lemma 6.5. The function in the top row is certainly continuous, because multiplication and inversion are continuous functions of the topological group $\mathrm{Homeo}^+(S^1)$. Restricting to subspaces in the second row, we see that the

$$
\begin{array}{ccc}
\mathrm{Homeo}^+(S^1) \times \mathrm{Homeo}^+(S^1) & \xrightarrow{(G,F) \mapsto F \circ G^{-1}} & \mathrm{Homeo}^+(S^1) \\
\uparrow & & \uparrow \\
\mathrm{N}(\Gamma) \times \widetilde{\mathcal{T}}(p) & \xrightarrow{(G,F) \mapsto F \circ G^{-1}} & \widetilde{\mathcal{T}}(p) \\
\pi_N \times \pi_p \big\downarrow & & \big\downarrow \pi_p \\
\mathrm{N}(\Gamma)/\Gamma \times \mathcal{T}(p) & \xrightarrow{([\partial\tilde{\psi}], [\partial\tilde{f}]) \mapsto [(\partial\tilde{f}) \circ (\partial\tilde{\psi})^{-1}]} & \mathcal{T}(p) \\
\Psi_p \times \Phi_p \big\uparrow \cong & & \cong \big\downarrow \Phi_p^{-1} \\
\mathrm{MCG}(S) \times \mathcal{T}(S) & \xrightarrow[A]{([\psi], [X,f]) \mapsto [X, f \circ \psi^{-1}]} & \mathcal{T}(S)
\end{array}
$$

FIGURE 7. Continuity of the action function A.

function in the second row is continuous as well. The quotient maps $\pi_N\colon \mathrm{N}(\Gamma) \to \mathrm{N}(\Gamma)/\Gamma$ and $\pi_p\colon \widetilde{\mathcal{T}}(p) \to \mathcal{T}(p)$ are open, because they are quotients by the group actions of Γ and $\mathrm{PSL}(2,\mathbb{R})$ respectively. Therefore the product map $\pi_N \times \pi_p$ is also open and surjective, hence a quotient map. By the universal property of quotients, the map in the second row descends to the function in the third row which is therefore continuous, as required. Thus the action function A is continuous. □

Corollary 6.7. *The mapping class group acts on the marked moduli space by homeomorphisms.*

Proof. Recall from equation (2) that for each fixed mapping class $[\psi] \in \mathrm{MCG}(S)$, its action on the marked moduli space is given by the function $A_{[\psi]}\colon \mathcal{T}(S) \to \mathcal{T}(S)$ which is $A_{[\psi]}[X,f] = A([\psi],[X,f])$. Then $A_{[\psi]}$ is continuous by Theorem 6.6, and $A_{[\psi^{-1}]}$ is evidently a continuous inverse, so $A_{[\psi]}$ is a homeomorphism. □

7. Embedding the Marked Moduli Space into the Character Space

In this section, we prove that the marked moduli space $\mathcal{T}(S)$ embeds into the $\mathrm{PSL}(2,\mathbb{R})$-character space $X(\pi_1(S), \mathrm{PSL}(2,\mathbb{R})) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ of the fundamental group. Here the $\mathrm{PSL}(2,\mathbb{R})$-representation space $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$ has the topology of pointwise convergence. The quotient is by the conjugation action of $\mathrm{PSL}(2,\mathbb{R})$, and a *character* is simply a conjugacy class of representations. The character space $X(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$ has the quotient topology. The embedding will imply that in case the surface S is of finite type, the topology on $\mathcal{T}(S)$ coincides with the usual topology on Teichmüller space. The marked moduli space injects into the character space in the obvious way via the character of a holonomy representation. We describe the injection Φ_{at} as follows. First fix a basepoint $s \in S$.

Definition 7.1 (The function Φ_{at})**.** *Suppose $[X,f] \in \mathcal{T}(S)$ is a marked hyperbolic structure. Take the basepoint on X to be $x = f(s)$, so that the marking map $f\colon S \to X$ induces an isomorphism of fundamental groups $f_*\colon \pi_1(S,s) \to \pi_1(X,x)$. Choose a pointed universal cover $p_X\colon (\mathbb{H}^2, \tilde{x}) \to (X,x)$ with deck group $\Gamma_X \subset \mathrm{PSL}(2,\mathbb{R})$, and let $\varphi_X\colon \pi_1(X,x) \xrightarrow{\cong}$*

$\Gamma_X \hookrightarrow \mathrm{PSL}(2,\mathbb{R})$ *be the corresponding holonomy representation. Composing the two group morphisms yields the representation* $\rho_{[X,f]} := \varphi_X \circ f_*\colon \pi_1(S,s) \to \mathrm{PSL}(2,\mathbb{R})$. *We define* $\Phi_{\mathrm{at}}[X,f]$ *to be the character* $[\rho_{[X,f]}]$ *in* $X(\pi_1(S,s), \mathrm{PSL}(2,\mathbb{R}))$.

Lemma 7.2. *For each* $[X,f] \in \mathcal{T}(S)$, $\Phi_{\mathrm{at}}[X,f]$ *is a well defined character of* $\mathrm{PSL}(2,\mathbb{R})$-*representations of* $\pi_1(S,s)$. *Further,* $\Phi_{\mathrm{at}}\colon \mathcal{T}(S) \to X(\pi_1(S,s), \mathrm{PSL}(2,\mathbb{R}))$ *is an injective function.*

Proof. First, we have to show that the character $[\rho_{[X,f]}]$ is independent of two choices made in the definition of Ψ_{at}, namely, the choice of the representative (X,f) of the marked hyperbolic structure, and the choice of the pointed universal cover p_X. We treat these one at a time, and in reverse order.

(1) The choice of the pointed universal cover $p_X\colon (\mathbb{H}^2, \tilde{x}) \to (X,x)$: A different pointed universal cover is of the form $p_X \circ \sigma^{-1}\colon (\mathbb{H}^2, \sigma(\tilde{x})) \to (X,x)$ for some $\sigma \in \mathrm{PSL}(2,\mathbb{R})$. Let $\rho'_{[X,f]}$ be the representation obtained using the new pointed universal cover. Suppose γ is a loop in S based at s, so that $[\gamma] \in \pi_1(S,s)$. Suppose that the lift of $f(\gamma)$ to the universal cover, with respect the covering map p_X, starting at $\tilde{x}$, has endpoint $\tilde{x}'$, so that the deck transformation $\rho_{[X,f]}[\gamma]$ maps $\tilde{x}$ to $\tilde{x}'$. Then the lift of the same curve, with respect to the covering map $p_X \circ \sigma^{-1}$, starting at $\sigma(\tilde{x})$ has endpoint $\sigma(\tilde{x}')$. Consequently, the deck transformation sending $\sigma(\tilde{x})$ to $\sigma(\tilde{x}')$ is $\sigma \circ \rho_{[X,f]}[\gamma] \circ \sigma^{-1}$. But this deck transformation is precisely the holonomy $\rho'_{[X,f]}[\gamma]$ with respect to the new pointed universal cover, so $\rho'_{[X,f]}[\gamma] = \sigma \circ \rho_{[X,f]}[\gamma] \circ \sigma^{-1}$. In other words, since this is true for all $[\gamma] \in \pi_1(S,s)$, $\rho'_{[X,f]}$ is the conjugate of $\rho_{[X,f]}$ by $\sigma \in \mathrm{PSL}(2,\mathbb{R})$, and so $[\rho'_{[X,f]}] = [\rho_{[X,f]}]$.

(2) The choice of the representative (X,f) of the marked hyperbolic structure: Suppose that (Y,g) is another representative of the same marked hyperbolic structure. The choice of a pointed universal cover $p_Y\colon (\mathbb{H}^2, \tilde{y}) \to (Y,y)$, where $y = g(s)$, leads to a holonomy representation $\varphi_Y\colon \pi_1(Y,y) \to \Gamma_Y \hookrightarrow \mathrm{PSL}(2,\mathbb{R})$. The equality $[X,f] = [Y,g]$ means that there exists an isometry $\psi\colon X \to Y$ such that g is homotopic to $\psi \circ f$. This homotopy lifts to universal covers as a homotopy from $\tilde{g}$ to $\tilde{\psi} \circ \tilde{f}$, where $\tilde{\psi}$, ψ $\tilde{f}$ and $\tilde{g}$ are lifts of ψ, f and g respectively, and let $\tilde{s} \in \mathbb{H}^2$ be such that $g(\tilde{s}) = \tilde{y}$. Let the endpoint of the track of $\tilde{s}$ under the lifted homotopy be $\tilde{y}'$, and let $y' = p_Y(\tilde{y}') = \psi \circ f(s)$. The pointed universal cover $p : (\mathbb{H}^2, \tilde{y}') \to (Y, y')$ leads to a holonomy representation $\varphi'_Y\colon \pi_1(Y,y') \to \Gamma_Y \hookrightarrow \mathrm{PSL}(2,\mathbb{R})$. For every closed curve γ in S based at s, $g(\gamma)$ is homotopic to $\psi \circ f(\gamma)$ in Y. Since the holonomies around homotopic curves are equal, $\varphi_Y(g_*[\gamma]) = \varphi'_Y((\psi \circ f)_*[\gamma])$ for every $[\gamma] \in \pi_1(S,s)$. Thus for every $[\gamma] \in \pi_1(S,s)$, we compute $\rho_{[Y,g]}[\gamma] = \varphi_Y(g_*[\gamma]) = \varphi'_Y((\psi \circ f)_*[\gamma]) = \varphi'_Y(\psi_*(f_*[\gamma])) = \psi_*(\varphi_X(f_*[\gamma])) = \psi_*(\rho_{[X,f]}[\gamma])$. Note that in the fourth equality, ψ_* denotes the π_1 functor on the left and the isometry between deck groups induced by $\tilde{\psi}$ on the right. However, $\psi_*\colon \Gamma_X \to \Gamma_Y$ is conjugation by $\tilde{\psi}$, so $\rho_{[Y,g]}[\gamma] = \psi_*(\rho_{[X,f]}[\gamma]) = \tilde{\psi} \circ \rho_{[X,f]}[\gamma] \circ \tilde{\psi}^{-1}$. In words, $\rho_{[Y,g]}[\gamma]$ is the conjugate of $\rho_{[X,f]}[\gamma]$ by $\tilde{\psi} \in \mathrm{PSL}(2,\mathbb{R})$. This holds for all $[\gamma] \in \pi_1(S,s)$, so we conclude that the representation $\rho_{[Y,g]}$ is conjugate to $\rho_{[X,f]}$ by some element of $\mathrm{PSL}(2,\mathbb{R})$. That is, $[\rho_{[Y,g]}] = [\rho_{[X,f]}]$, so that $\Phi_{\mathrm{at}}[X,f]$ is a well defined character.

Finally, we show that Φ_{at} is injective. Suppose $[X,f], [Y,g] \in \mathcal{T}(S)$ are marked hyperbolic structures such that $[\rho_{[X,f]}] = [\rho_{[Y,g]}]$. This means that there is an element $\sigma \in \mathrm{PSL}(2,\mathbb{R})$

such that $\sigma \circ \rho_{[X,f]}[\gamma] \circ \sigma^{-1} = \rho_{[Y,g]}[\gamma]$ for every $[\gamma] \in \pi_1(S,s)$. Thus $\sigma \circ \Gamma_X \circ \sigma^{-1} = \Gamma_Y$, which we rearrange to $\sigma \circ \Gamma_X = \Gamma_Y \circ \sigma$, so σ is equivariant. Hence σ descends to an isometry $\psi\colon X \to Y$. Further, for every $[\gamma] \in \pi_1(S,s)$, we have $\varphi_Y \circ g_*[\gamma] = \rho_{[Y,g]}[\gamma] = \sigma \circ \rho_{[X,f]}[\gamma] \circ \sigma^{-1} = \psi_* \circ \varphi_X \circ f_*[\gamma] = \varphi_Y \circ \psi_* \circ f_*[\gamma]$. In the last equality, ψ_* denotes the isomorphism of deck groups on the left and the π_1 functor on the right. Since φ_Y is an isomorphism, $(\psi \circ f)_* = g_*$ as homomorphisms $\pi_1(S,s) \to \pi_1(Y,y)$. Since S is a $K(\pi_1(S,s),1)$ classifying space, we conclude that $\psi \circ f$ is homotopic to g. Therefore $[X,f] = [Y,g]$, and Φ_{at} is injective. □

Since the topology on $\mathcal{T}(S)$ is defined using a universal cover p and the function Φ_p, in order to prove Φ_{at} is an embedding, we must compute Φ_{at} in terms of Φ_p. To that end, fix a pointed universal cover $p\colon (\mathbb{H}^2, \tilde{s}) \to (S,s)$ with deck group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$, which is a torsion-free Fuchsian group of the first kind. This leads to a holonomy representation $\varphi\colon \pi_1(S,s) \to \Gamma \hookrightarrow \mathrm{PSL}(2,\mathbb{R})$ as before. For any marked hyperbolic structure $[X,f] \in \mathcal{T}(S)$ with a pointed universal cover $p_X\colon (\mathbb{H}^2, \tilde{x}) \to (X,x)$, where $x = f(s)$, the marking map f lifts to a map $\tilde{f}$, which extends, by Proposition 4.1(1), to a homeomorphism at infinity $\partial\tilde{f}$ representing $\Phi_p[X,f]$. The rough idea is that $\Phi_{\mathrm{at}}[X,f]$ is the conjugate of φ by $\Phi_p[X,f]$. As before, we have the representation $\rho_{[X,f]} = \varphi_X \circ f_*$. This is related to $\partial\tilde{f}$ and φ as follows.

Proposition 7.3 (Φ_{at} in terms of Φ_p)**.** *For any $[\gamma] \in \pi_1(S,s)$, we have*

$$\rho_{[X,f]}[\gamma] = (\partial\tilde{f}) \circ \varphi[\gamma] \circ (\partial\tilde{f})^{-1} \tag{7}$$

In general, if $\Phi_p[X,f]$ is represented by the homeomorphism F, then $\Phi_{\mathrm{at}}[X,f]$ is represented by the representation R which satisfies, for all $[\gamma] \in \pi_1(S,s)$,

$$R[\gamma] = F \circ \varphi[\gamma] \circ F^{-1} \tag{8}$$

Proof. We compute $\rho_{[X,f]}[\gamma] = \varphi_X(f_*[\gamma]) = f_*(\varphi[\gamma]) = (\partial\tilde{f}) \circ \varphi[\gamma] \circ (\partial\tilde{f})^{-1}$ by Proposition 4.1(3). Note that in the second equality, the f_* on the left hand side is the π_1 functor, whereas on the right hand side, f_* is the isomorphism between the deck groups Γ and Γ_X induced by $\tilde{f}$. Any other homeomorphism F representing $\Phi_p[X,f]$ is of the form $\sigma \circ (\partial\tilde{f})$ for some $\sigma \in \mathrm{PSL}(2,\mathbb{R})$. Then for all $[\gamma] \in \pi_1(S)$, we have $\sigma \circ \rho_{[X,f]}[\gamma] \circ \sigma^{-1} = \sigma \circ (\partial\tilde{f}) \circ \varphi[\gamma] \circ (\partial\tilde{f})^{-1} \circ \sigma^{-1} = (\sigma \circ (\partial\tilde{f})) \circ \varphi[\gamma] \circ (\sigma \circ (\partial\tilde{f}))^{-1} = F \circ \varphi[\gamma] \circ F^{-1} = R[\gamma]$. Thus the representation R is conjugate to the representation $\rho_{[X,f]}$, and hence the characters of $\rho_{[X,f]}$ and R are equal. That is, R represents $\Phi_{\mathrm{at}}[X,f]$ and satisfies equation (8). □

We can also compute Φ_p in terms of Φ_{at}, by computing the image of the homeomorphism at infinity on the sinks of hyperbolic elements. Call an element $[\gamma] \in \pi_1(S,s)$ *hyperbolic* if $\varphi[\gamma]$ is a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$. In this case, $R[\gamma] = f_*(\varphi[\gamma])$ is also hyperbolic by Lemma 4.4 and the fact that conjugates in $\mathrm{PSL}(2,\mathbb{R})$ have the same type. Further, a triple $([\gamma_1],[\gamma_2],[\gamma_3])$ of hyperbolic elements in $\pi_1(S,s)$ is *positively oriented* if the triple $((\varphi[\gamma_1])_\infty, (\varphi[\gamma_2])_\infty, (\varphi[\gamma_3])_\infty)$ consists of distinct points in S^1 and is positively oriented. In this case, the triple $((R[\gamma_1])_\infty, (R[\gamma_2])_\infty, (R[\gamma_3])_\infty)$ is also positively oriented by equation (8). Thus the notions of hyperbolicity and positively oriented triple do not depend on the choice of the universal cover p, as long as its deck group Γ is of the first kind. The rough idea is that $\Phi_p[X,f]$ maps sinks of the φ representation of hyperbolic elements of $\pi_1(S,s)$ to the sinks of their $\Phi_{\mathrm{at}}[X,f]$ representation.

Proposition 7.4 (Φ_p in terms of Φ_{at})**.** *For any hyperbolic $[\gamma] \in \pi_1(S,s)$, we have*

$$\partial\tilde{f}((\varphi[\gamma])_\infty) = (\rho_{[X,f]}[\gamma])_\infty \tag{9}$$

If general, if $\Phi_{\text{at}}[X,f]$ is represented by the representation R, then $\Phi_p[X,f]$ is represented by the homeomorphism F which satisfies, for all $[\gamma] \in \pi_1(S,s)$,

$$F((\varphi[\gamma])_\infty) = (R[\gamma])_\infty \tag{10}$$

Proof. To establish equation (9), we use Definition 4.5 and compute that for hyperbolic $[\gamma] \in \pi_1(S,s)$, $\partial\tilde{f}((\varphi[\gamma])_\infty) = (f_*(\varphi[\gamma]))_\infty = (\varphi_X(f_*[\gamma]))_\infty = (\rho_{[X,f]}[\gamma])_\infty$. Note that in the second equality, the f_* on the left hand side is the isomorphism between the deck group Γ and Γ_X induced by $\tilde{f}$, whereas on the right hand side, f_* is the π_1 functor. In the general case, since $\Phi_{\text{at}}[X,f]$ is represented by R, it follows that R is conjugate to the representation $\rho_{[X,f]}$ by some $\sigma \in \text{PSL}(2,\mathbb{R})$. That is, for every hyperbolic $[\gamma] \in \pi_1(S,s)$, we have $R[\gamma] = \sigma \circ \rho_{[X,f]}[\gamma] \circ \sigma^{-1}$. Therefore the sink $(R[\gamma])_\infty$ is just the image under σ of the sink $(\rho_{[X,f]}[\gamma])_\infty$. In other words, $(R[\gamma])_\infty = \sigma((\rho_{[X,f]}[\gamma])_\infty)$, which equals, by equation (9), $\sigma \circ (\partial\tilde{f})((\varphi[\gamma])_\infty)$. Setting $F = \sigma \circ (\partial\tilde{f})$, we see that F also represents the same right coset of $\text{PSL}(2,\mathbb{R})$ in $\text{Homeo}^+(S^1)$ as that of $(\partial\tilde{f})$, and hence represents $\Phi_p[X,f]$. Thus we have established equation (10). □

Theorem 7.5. *$\Phi_{\text{at}}\colon \mathcal{T}(S) \to X(\pi_1(S,s), \text{PSL}(2,\mathbb{R}))$ is a topological embedding.*

Proof. We need to show that a sequence of marked hyperbolic structures $[X_i, f_i]$ converges to $[X,f]$ in $\mathcal{T}(S)$ as $i \to \infty$ if and only if the corresponding sequence of characters $\Phi_{\text{at}}[X_i, f_i]$ converges to $\Phi_{\text{at}}[X,f]$ in $X(\pi_1(S,s), \text{PSL}(2,\mathbb{R}))$ as $i \to \infty$. As before, fix a universal over $p\colon \mathbb{H}^2 \to S$ with deck group $\Gamma \subset \text{PSL}(2,\mathbb{R})$, a torsion-free Fuchsian group of the first kind. By Definition 5.5, Φ_p is a homeomorphism and so the convergence $[X_i, f_i] \to [X,f]$ in $\mathcal{T}(S)$ as $i \to \infty$ is equivalent to the convergence $\Phi_p[X_i, f_i] \to \Phi_p[X,f]$ in $\mathcal{T}(p)$ as $i \to \infty$.

We begin with the 'only if' part. Suppose $\Phi_p[X_i, f_i] \to \Phi_p[X,f]$ in $\mathcal{T}(p)$ as $i \to \infty$. We can promote this to convergence in $\widetilde{\mathcal{T}}(p)$ by constructing a continuous section $\Sigma_1\colon \mathcal{T}(p) \to \widetilde{\mathcal{T}}(p)$. For any $[\hat{F}] \in \mathcal{T}(p)$, let $\Sigma_1([\hat{F}])$ be the unique homeomorphism of S^1 in the right coset $\text{PSL}(2,\mathbb{R}) \circ \hat{F}$ that fixes the three points $0, 1, \infty$. In fact, $\Sigma_1(\hat{F}) = M(\hat{F}(0), \hat{F}(1), \hat{F}(\infty))^{-1} \circ \hat{F}$, where $M(a,b,c)$ is the Möbius transformation mapping the triple $(0,1,\infty)$ to the triple (a,b,c). Since $M(a,b,c)$ is a continuous function of $a,b,c \in S^1$, evaluations at $0,1,\infty$ are continuous functions of $\hat{F}$, and compositions are continuous, we conclude that Σ_1 is a continuous function. Therefore $\Sigma_1(\Phi_p[X_i, f_i]) \to \Sigma_1(\Phi_p[X,f])$ as $i \to \infty$. Denoting $\Sigma_1(\Phi_p[X_i, f_i])$ by F_i and $\Sigma_1(\Phi_p[X,f])$ by F, we have $F_i \to F$ in $\widetilde{\mathcal{T}}(p)$ as $i \to \infty$.

Note that, by Proposition 7.3, $\Phi_{\text{at}}[X,f]$ is represented by the representation ρ which satisfies, for each $[\gamma] \in \pi_1(S,s)$, the relation $\rho[\gamma] = F \circ \varphi[\gamma] \circ F^{-1}$. Similarly, $\Phi_{\text{at}}[X_i, f_i]$ is represented by the representation ρ_i which satisfies, for each $[\gamma] \in \pi_1(S,s)$, the relation $\rho_i[\gamma] = (F_i) \circ \varphi[\gamma] \circ (F_i)^{-1}$. This $\rho_i[\gamma]$ converges, as $i \to \infty$, to $F \circ \varphi[\gamma] \circ F^{-1} = \rho[\gamma]$. Here we are using the fact that $\text{PSL}(2,\mathbb{R})$ is an embedded subspace of $\text{Homeo}^+(S^1)$ On the level of representations, this means $\rho_i \to \rho$ and on the level of characters, $[\rho_i] \to [\rho]$ as $i \to \infty$. In other words, $\Phi_{\text{at}}[X_i, f_i] \to \Phi_{\text{at}}[X,f]$ in $X(\pi_1(S,s), \text{PSL}(2,\mathbb{R}))$ as $i \to \infty$ and therefore Φ_{at} is continuous.

Next we prove the 'if' part. Suppose $\Phi_{\text{at}}[X_i, f_i] \to \Phi_{\text{at}}[X,f]$ in $X(\pi_1(S,s), \text{PSL}(2,\mathbb{R}))$ as $i \to \infty$. We promote the convergence of characters to convergence of representations by constructing a continuous section $\Sigma_2\colon \Phi_{\text{at}}(\mathcal{T}(S)) \to \text{Hom}(\pi_1(S,s), \text{PSL}(2,\mathbb{R}))$. Let $(\alpha_1, \alpha_2, \alpha_3)$

be a positively oriented triple of hyperbolic elements of $\pi_1(S,s)$, which exists since the set Γ_∞ is dense in S^1 due to Proposition 3.1. Now, for any $[\hat{\rho}] \in \Phi_{\mathrm{at}}(\mathcal{T}(S))$, let $\Sigma_2[\hat{\rho}]$ be the unique representation that represents the same character as $\hat{\rho}$ and such that the sinks of $\Sigma_2[\hat{\rho}](\alpha_1), \Sigma_2[\hat{\rho}](\alpha_2), \Sigma_2[\hat{\rho}](\alpha_3)$ are $0, 1, \infty$ respectively. In fact $\Sigma_2[\hat{\rho}]$ is the conjugate of $\hat{\rho}$ by the Möbius transformation $M((\hat{\rho}(\alpha_1))_\infty, (\hat{\rho}(\alpha_2))_\infty, (\hat{\rho}(\alpha_3))_\infty)^{-1}$. Since $M(a,b,c)$ is a continuous function of $a, b, c \in S^1$, evaluations at $\alpha_1, \alpha_2, \alpha_3 \in \pi_1(S,s)$ are continuous functions of the representation $\hat{\rho}$, the sink of a hyperbolic element is a continuous function of the hyperbolic element, and compositions are continuous, we conclude that Σ_2 is continuous. Therefore $\Sigma_2(\Phi_{\mathrm{at}}[X_i, f_i]) \to \Sigma_2(\Phi_{\mathrm{at}}[X, f])$ as $i \to \infty$. Denoting $\Sigma_2(\Phi_{\mathrm{at}}[X_i, f_i])$ by ρ_i and $\Sigma_2(\Phi_{\mathrm{at}}[X, f])$ by ρ, we have $\rho_i \to \rho$ in $\mathrm{Hom}(\pi_1(S,s), \mathrm{PSL}(2,\mathbb{R}))$ as $i \to \infty$. In particular, this means that for all hyperbolic elements $[\gamma] \in \pi_1(S,s)$, we have $\rho_i[\gamma] \to \rho[\gamma]$, and so $(\rho_i[\gamma])_\infty \to (\rho[\gamma])_\infty$ as $i \to \infty$.

Note that, by Proposition 7.4, $\Phi_p[X, f]$ is represented by the homeomorphism F of S^1, which satisfies, for every hyperbolic $[\gamma] \in \pi_1(S,s)$, the relation $F((\varphi[\gamma])_\infty) = (\rho[\gamma])_\infty$. Similarly $\Phi_p[X_i, f_i]$ is represented by the homeomorphism F_i of S^1 which satisfies, for every hyperbolic $[\gamma] \in \pi_1(S,s)$, the relation $F_i((\varphi[\gamma])_\infty) = (\rho_i[\gamma])_\infty$. To show that $[X_i, f_i]$ converges to $[X, f]$, it is enough to prove that F_i converges to F as $i \to \infty$, in the topology of uniform convergence on S^1 with respect to some path metric d, such as the visual metric based at $\tilde{s} \in \mathbb{H}^2$. This would imply that as $i \to \infty$, $\Phi_p[X_i, f_i] = [F_i]$ converges to $\Phi_p[X, f] = [F]$. Let $\varepsilon > 0$ be given. Reducing ε if necessary, assume ε is less than half the d-length of S^1. We choose a finite set of points in $(\Gamma_X)_\infty$ which is $\frac{\varepsilon}{12}$-dense in S^1. This is possible because S^1 is compact and $(\Gamma_X)_\infty$ is dense in S^1 due to Proposition 3.1. Represent these points as sinks $(\rho[\gamma_1])_\infty, (\rho[\gamma_2])_\infty, \ldots, (\rho[\gamma_n])_\infty$ of holonomies around hyperbolic $[\gamma_1], [\gamma_2], \ldots, [\gamma_n] \in \pi_1(S,s)$. Relabelling if necessary, and removing duplicates, assume that $(\rho[\gamma_1])_\infty, (\rho[\gamma_2])_\infty, \ldots, (\rho[\gamma_n])_\infty$ are distinct and in positive circular order. These points divide the circle at infinity into n intervals, each of length less than $\frac{\varepsilon}{6}$. Now for each $j = 1, 2, \ldots, n$, we have $(\varphi[\gamma_j])_\infty = F^{-1}(\rho[\gamma_j])_\infty$. As F^{-1} is an orientation preserving homeomorphism, it follows that the sinks $(\varphi[\gamma_1])_\infty, (\varphi[\gamma_2])_\infty, \ldots, (\varphi[\gamma_n])_\infty$ are in positive circular order. Similarly for each i and each $j = 1, 2, \ldots, n$, we have $(\rho_i[\gamma_j])_\infty = F_i((\varphi[\gamma_j])_\infty)$, and since F_i is an orientation preserving homeomorphism, it follows that the points $(\rho_i[\gamma_1])_\infty, (\rho_i[\gamma_2])_\infty, \ldots, (\rho_i[\gamma_n])_\infty$ are also in positive circular order. Since the sink of a hyperbolic element is a continuous function of the hyperbolic element, for each $j = 1, 2, \ldots, n$, the convergence $\rho_i[\gamma_j] \to \rho[\gamma_j]$ implies $(\rho_i[\gamma_j])_\infty \to (\rho[\gamma_j])_\infty$ as $i \to \infty$. Therefore for all sufficiently large i, namely those larger than some index N, and for each of the finitely many indices $j = 1, 2, \ldots, n$, we have $d((\rho_i[\gamma_j])_\infty, (\rho[\gamma_j])_\infty) < \frac{\varepsilon}{6}$. Further, for $i > N$ and $j = 1, 2, \ldots, n$, we have $d((\rho_i[\gamma_j])_\infty, (\rho_i[\gamma_{j+1}])_\infty) \leq d((\rho_i[\gamma_j])_\infty, (\rho[\gamma_j])_\infty) + d((\rho[\gamma_j])_\infty, (\rho[\gamma_{j+1}])_\infty) + d((\rho[\gamma_{j+1}])_\infty, (\rho_i[\gamma_{j+1}])_\infty) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}$ (subscripts of the letter γ being modulo n). Thus for $i > N$, the points $(\rho_i[\gamma_1])_\infty, (\rho_i[\gamma_2])_\infty, \ldots, (\rho_i[\gamma_n])_\infty$ divide S^1 into intervals of length less than $\frac{\varepsilon}{2}$.

Suppose $q \in S^1$ is an arbitrary point and $i > N$ be an arbitrary index. Then for some j, q belongs to an interval bounded by points $(\varphi[\gamma_j])_\infty, (\varphi[\gamma_{j+1}])_\infty$. Since F_i is an orientation preserving homeomorphism, it follows that $F_i(q)$ is in the interval bounded by the points $(\rho_i[\gamma_j])_\infty$ and $(\rho_i[\gamma_{j+1}])_\infty$. Therefore $d(F_i(q), (\rho_i[\gamma_j])_\infty)$ is less than the length of this interval, which is less than $\frac{\varepsilon}{2}$. Similarly, F is an orientation preserving homeomorphism, so $F(q)$ lies in the interval bounded by the points $(\rho[\gamma_j])_\infty$ and $(\rho[\gamma_{j+1}])_\infty$, and hence $d(F(q), (\rho[\gamma_j])_\infty)$ is less than the length of this interval, which is less than $\frac{\varepsilon}{6}$. Now we have $d(F_i(q), F(q)) \leq$

$d(F_i(q), (\rho_i[\gamma_j])_\infty) + d((\rho_i[\gamma_j])_\infty, (\rho[\gamma_j])_\infty) + d((\rho[\gamma_j])_\infty, F(q)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} < \varepsilon$. Since $q \in S^1$ was arbitrary, for every $i > N$, the maps F_i and F are uniformly ε-close. Since ε was an arbitrary positive number, we infer that $F_i \to F$ as $i \to \infty$ in the topology of uniform convergence. This concludes the 'if' part, that is, $\Phi_{\mathrm{at}}^{-1} \colon \Phi_{\mathrm{at}}(\mathcal{T}(S)) \to \mathcal{T}(S)$ is continuous. Hence Φ_{at} is an embedding. □

Corollary 7.6. *If S is a finite type surface of negative Euler characteristic, then the topology on $\mathcal{T}(S)$ agrees with the topology on Teichmüller space.*

Proof. Indeed, one of the ways to describe the Teichmüller space is as the character space of all discrete faithful representations of $\pi_1(S, s)$ into $\mathrm{PSL}(2, \mathbb{R})$ (see [FM11, Chapter 10]). In particular, the topology on the Teichmüller space is defined as a subspace of the full character space $X(\pi_1(S, s), \mathrm{PSL}(2, \mathbb{R}))$. But this is exactly the topology on $\mathcal{T}(S)$ that we have defined above. Thus $\mathcal{T}(S)$ reduces to the usual Teichmüller space in case S is a finite type surface. □

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