Math 4310

Homework 10

Due **11/30/2012**

Name: ______________________________

Collaborators: ______________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________/

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Vector space	/4	/4
Example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Let $\mathbb{F} = \mathbb{C}$, and let $m, n \in \mathbb{N}$ with $0 < m \leq n$. Prove that there exists a polynormial $p(x) \in \mathbb{F}[x]$ of degree n with exactly m distinct roots. What goes wrong if $\mathbb{F}$ is a finite field?

2. Suppose that $p(x) \in \mathbb{C}[x]$ is of degree m. Show that $p(x)$ has m distinct roots if and only if $p(x)$ and $p'(x)$ have no roots in common. (Here, $p'(x)$ is the derivative of $p(x)$. Just as for polynomials with real coefficients, $\frac{d}{dx}[\lambda x^n] = n\lambda x^{n-1}$, for $n \geq 0$ and $\lambda \in \mathbb{C}$.)

3. Let V be a vector space over $\mathbb{F}$, and $U_1, \ldots, U_k$ subspaces of V. Let $T : V \to V$ be a linear transformation. Show that if $U_1, \ldots, U_k$ are invariant under T, then their sum $U_1 + \cdots + U_k$ is also invariant under T.

4. Consider the linear transformation

$$\begin{aligned} T : \mathbb{R}^3 &\to \mathbb{R}^3 \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\mapsto \begin{pmatrix} 0 \\ 3x_1 \\ 5x_3 \end{pmatrix}. \end{aligned}$$

Find all the eigenvalues and eigenvectors of T.

5. Consider the linear transformation

$$\begin{aligned} T : \mathbb{F}^n &\to \mathbb{F}^n \\ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} &\mapsto \begin{pmatrix} x_1 + \cdots + x_n \\ x_1 + \cdots + x_n \\ \vdots \\ x_1 + \cdots + x_n \end{pmatrix}. \end{aligned}$$

   Find all the eigenvalues and eigenvectors of T.

6. Let V be a finite dimensional vector space, and $S, T \in \mathscr{L}(V, V)$. Prove that $S \circ T$ and $T \circ S$ have the same eigenvalues.

7. Give an example of a linear transformation $T \in \mathscr{L}(\mathbb{R}^4, \mathbb{R}^4)$ with no real eigenvalues. Explain why there are no eigenvalues.

8. Let V be a finite dimensional inner product space, and U a subspace. Let $P_U : V \to V$ denote the orthogonal projection onto U. What are the eigenvalues of P_U? Find an orthonormal basis of eigenvectors.

Extended Glossary.
There is no extended glossary in this last HW assignment.