1. Let V be a vector space, and X and Y complementary subspaces so that $V = X \oplus Y$. Set $P_1 = \mathrm{Proj}_{(X,Y)}$ and $P_2 = \mathrm{Proj}_{(Y,X)}$. Prove the following.

   (a) $P_1 \circ P_1 = P_1$
   For v in V, there is a unique $x \in X$ and $y \in Y$ such that $v = x + y$. Then

   $$(P_1 \circ P_1)(v) = P_1(P_1(v)) = P_1(P_1(x+y)) = P_1(x) = x = P_1(x+y) = P_1(v).$$

   (b) $P_1 + P_2 = \mathbb{1}_V$
   For v in V, there is a unique $x \in X$ and $y \in Y$ such that $v = x + y$. Then

   $$(P_1 + P_2)(v) = P_1(v) + P_2(v) = P_1(x+y) + P_2(x+y) = x + y = v = \mathbb{1}_V(v).$$

   (c) $P_1 \circ P_2 = \mathbb{0}_{\mathbb{V}\to\mathbb{V}}$ is the zero transformation
   For v in V, there is a unique $x \in X$ and $y \in Y$ such that $v = x + y$. Then

   $$(P_1 \circ P_2)(v) = P_1(P_2(v)) = P_1(P_2(x+y)) = P_1(y) = \mathbb{0} = \mathbb{0}_{V\to V}(v).$$

2. Let V and W be vector spaces over $\mathbb{F}$, and let $\mathscr{L}(V, W)$ denote the linear transformations from V to W. We proved in class that $\mathscr{L}(V, W)$ is a subspace of $\mathscr{F}un(V, W)$. Suppose that $\dim(V) = 2$, $\dim(W) = 3$, and $v_1, v_2 \in V$ and $w_1, w_2, w_3 \in W$ are bases. Find a basis for $\mathscr{L}(V, W)$.

   Suppose that $\dim(V) = 2, \dim(W) = 3$, and $v_1, v_2 \in V$ and $w_1, w_2, w_3 \in W$ are bases. For $j \in \{1, 2\}, i \in \{1, 2, 3\}$, let $f_{ij} : V \to W$ be the linear transformation defined by

   $$f_{ij}(v_k) = \begin{cases} w_i & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}.$$

   We claim that the f_{ij} form a basis of $\mathscr{L}(V, W)$.

   First we will show linear independence. Suppose

   $$\sum_{j=1}^{2} \sum_{i=1}^{3} \alpha_{ij} f_{ij}(x) = \mathbb{0}(x).$$

   Then in particular, for $k \in \{1, 2\}$,

   $$\sum_{j=1}^{2} \sum_{i=1}^{3} \alpha_{ij} f_{ij}(v_k) = \mathbb{0}(v_k)$$
   $$\sum_{i=1}^{3} \alpha_{i,k} w_i = 0.$$

   Since $\{w_1, w_2, w_3\}$ is linearly independent, this forces $\alpha_{i,k} = 0$ for each i and k, giving us linear independence, as desired.

Now let f be any element of $\mathscr{L}(V, W)$. Then $f(v_j)$ is a linear combination of $\{w_1, w_2, w_3\}$, say

$$f(v_j) = \sum_{i=1}^{3} \alpha_{ij} w_i.$$

Then

$$f = \sum_{j=1}^{2} \sum_{i=1}^{3} \alpha_{ij} f_{ij}$$

and the f_{ij} span $\mathscr{L}(V, W)$, completing the proof.

3. We will let $\mathscr{L}(V)$ denote all linear transformations from a vector space V to itself (sometimes called **linear operators**). Let $T \in \mathscr{L}(V)$, and let T^2 denote the composition $T \circ T$.

   (a) Prove that if $T^2 = T$, then $V = \ker(T) \oplus \ker(T - \mathbb{1}_V)$.
   (There are interesting such T: *e.g.* P_1 in 1(a).)
   Since $T^2 = T$, this can be rewritten as $(T^2 - T)(v) = 0$ for all $v \in V$. To show that $V = \ker(T) \oplus \ker(T - \mathbb{1}_V)$, we need to show $V = \ker(T) + \ker(T - \mathbb{1}_V)$ and $\ker(T) \cap \ker(T - \mathbb{1}_V) = \{0\}$.
   To show that $V = \ker(T) + \ker(T - \mathbb{1}_V)$, let $v \in V$. We claim that $v = (v - T(v)) + (T(v))$ is a decomposition such that $(v - T(v)) \in \ker(T)$ and $T(v) \in \ker(T - \mathbb{1}_V)$. Indeed, since

   $$T(v - T(v)) = T((\mathbb{1}_V - T)(v)) = (T - T^2)(v) = 0,$$

   we have $(v - T(v)) \in \ker(T)$. Moreover, since

   $$(T - \mathbb{1}_V)(T(v)) = (T^2 - T)(v) = 0,$$

   we have $T(v) \in \ker(T - \mathbb{1}_V)$. This verifies our claim.
   To show that $\ker(T) \cap \ker(T - \mathbb{1}_V) = \{0\}$, let $v \in \ker(T) \cap \ker(T - \mathbb{1}_V)$. This implies $T(v) = 0$ and $(T - \mathbb{1}_V)(v) = T(v) - v = 0$, which further implies $v = 0$. Therefore, $\ker(T) \cap \ker(T - \mathbb{1}_V) = \{0\}$.

   (b) Prove that if $V = \ker(T) + \ker(T - \mathbb{1}_V)$, then $T^2 = T$.
   Assume $V = \ker(T) + \ker(T - \mathbb{1}_V)$. This means that given $v \in V$, we can write $v = u + w$ for some $u \in \ker(T)$ and $w \in \ker(T - \mathbb{1}_V)$. The condition $w \in \ker(T - \mathbb{1}_V)$ is equivalent to $T(w) = w$. It follows that

   $$T(v) = T(u + w) = T(u) + T(w) = 0 + w = w$$

   $$T^2(v) = T(T(v)) = T(w) = w,$$

   so $T^2 = T$.

   (c) Give an example of a vector space V and $T \in \mathscr{L}(V)$ such that $T^2 = -\mathbb{1}_V$.
   Let $V = \mathbb{C}$ with the underlying field being $\mathbb{R}$, and let $T : V \to V$ be the linear transformation defined by $T(v) = iv$. Then $T^2(v) = -v$.

(d) Prove that if $T^2 = \mathbb{0}_{V\to V}$ is the zero transformation, then $\operatorname{rank}(T) \le \frac{\dim(V)}{2}$.
The first isomorphism theorem tells us $V/\ker(T) \cong \operatorname{Im}(T)$, so $\dim V = \dim\ker(T) + \dim\operatorname{Im}(T)$. Since $T^2 = \mathbb{0}$, it must be that $\operatorname{Im}(T) \subseteq \ker(T)$, so $\dim\operatorname{Im}(T) \le \dim\ker(T)$. Putting these together, we get $\dim V \ge 2\dim\operatorname{Im}(T)$, or $\operatorname{rank}(T) = \dim\operatorname{Im}(T) \le \frac{\dim V}{2}$.

4. Let $\mathbb{F}$ be a field and $V = \mathscr{P}o\ell(\mathbb{F})$. Define $M : \mathscr{P}o\ell(\mathbb{F}) \to \mathscr{P}o\ell(\mathbb{F})$ by $M(p(x)) = x\cdot p(x)$. Show that M is a linear transformation. Find $\ker(M)$ and $Im(M)$. Is $M^n = M\circ\cdots\circ M = \mathbb{0}_{V\to V}$ the zero transformation for any n?

   Let $p(x), q(x) \in \mathscr{P}o\ell(\mathbb{F})$ and $c \in \mathbb{F}$. Then

$$M(p(x) + q(x)) = x\cdot(p(x) + q(x)) = x\cdot p(x) + x\cdot q(x) = M(p(x)) + M(q(x)),$$

$$M(c\cdot p(x)) = x\cdot(cp(x)) = c\cdot x\cdot p(x) = c\cdot M(p(x)),$$

   so M is a linear transformation.

   Because $\deg(M(p(x)) = \deg p(x) + 1$ and 0 has degree $-\infty$, $\ker M = \{0\}$. Equivalently, M is injective, so M^n can never be the zero transformation. (As an alternative solution for showing $M^n \ne \mathbb{0}$ for all n, notice that $M^n(1) = x^n \ne \mathbb{0}$ for all n.)

   Finally we claim $\operatorname{Im}(M) = \{p(x) \in \mathscr{P}o\ell(\mathbb{F}) : p(0) = 0\}$. Certainly if $p(x) \in \operatorname{Im}(M)$, then there is a $q(x) \in \mathscr{P}o\ell(\mathbb{F})$ such that $p(x) = x\cdot q(x)$, so $p(0) = 0\cdot q(0) = 0$. This shows that $\operatorname{Im} M \subseteq \{p(x) \in \mathscr{P}o\ell(\mathbb{F}) : p(0) = 0\}$. To show the other inequality, note that if $p(x) = \sum_{i=0}^{n} \alpha_i x^i$ and $p(0) = 0$, then $\alpha_0 = 0$, so

$$p(x) = \sum_{i=1}^{n} \alpha_i x^i = x\sum_{i=0}^{n-1} \alpha_{i+1} x^i = x\cdot q(x) = M(q(x)),$$

   proving our claim.

5. Let $T : V \to W$ be a linear transformation, and suppose that X_1 and X_2 are subspaces of V both containing $\ker(T)$. Prove that if $T(X_1) = T(X_2)$, then $X_1 = X_2$.

   First we will show $X_1 \subseteq X_2$. Suppose $x_1 \in X_1$. Because $T(X_1) = T(X_2)$, there is $x_2 \in X_2$ such that $T(x_1) = T(x_2)$. Since T is linear, this means $T(x_1 - x_2) = 0$, so $v = x_1 - x_2$ is in $\ker(T)$. X_2 contains $\ker(T)$ and X_2 is closed under addition, so $x_1 = v + x_2$ is in X_2, which shows $X_1 \subseteq X_2$. A similar argument shows $X_2 \subseteq X_1$, which concludes our proof.

6. Let U and V be vector spaces with respective subspaces X and Y. Prove that there is an isomorphism $(U \times V)/(X \times Y) \cong (U/X) \times (V/Y)$.

   Let

$$\begin{aligned} f : (U\times V)/(X\times Y) &\to (U/X)\times(V/Y) \\ [(u,v)]_{X\times Y} &\mapsto ([u]_X, [v]_Y). \end{aligned}$$

   To show that this is well-defined, suppose $[(u_1, v_1)]_{X\times Y} = [(u_2, v_2)]_{X\times Y}$. By definition of quotient spaces, there is some $(x, y) \in X\times Y$ such that $(u_1, v_1) = (u_2, v_2) + (x, y)$. This means $u_1 = u_2 + x$ and $v_1 = v_2 + y$, which shows $[u_1]_X = [u_2]_X$ and $[v_1]_Y = [v_2]_Y$. Thus

$$f([(u_1, v_1)]_{X\times Y}) = ([u_1]_X, [v_1]_Y) = ([u_2]_X, [v_2]_Y) = f([(u_2, v_2)]_{X\times Y}),$$

and f is well-defined, because it respects the equivalence relations.

Note that

$$\begin{aligned} f([(u_1,v_1)]_{X\times Y}+[(u_2,v_2)]_{X\times Y}) &= f([(u_1+u_2,v_1+v_2)]_{X\times Y}) = ([u_1+u_2]_X,[v_1+v_2]_Y) \\ &= ([u_1]_X+[u_2]_X,[v_1]_Y+[v_2]_Y) = ([u_1]_X,[v_1]_Y)+([u_2]_X,[v_2]_Y) \\ &= f([u_1,v_1]_{X\times Y})+f([u_2,v_2]_{X\times Y}) \end{aligned}$$

and

$$\begin{aligned} f(c[(u,v)]_{X\times Y}) &= f([(cu,cv)]_{X\times Y}) = ([cu]_X,[cv]_Y) \\ &= (c[u]_X,c[v]_Y) = c([u]_X,[v]_Y) = cf([(u,v)]_{X\times Y}), \end{aligned}$$

so f is a linear transformation.

Suppose $[(u,v)]_{X\times Y}$ is in $\ker(f)$. Then $f([(u,v)]_{X\times Y}) = ([0]_X,[0]_Y)$, which implies $([u]_X,[v]_Y) = ([0]_X,[0]_Y)$. This means $[u]_X = [0]_X$ and $[v]_Y = [0]_Y$, so $[(u,v)]_{X\times Y} = [0]_{X\times Y}$. Thus $\ker(f) = \{0\}$, so f is injective.

Finally for any $([u]_X,[v]_Y)$ in $U/X \times V/Y$, take u and v to be representative elements of $[u]_X$ and $[v]_Y$ respectively. Then $f([(u,v)]_{X\times Y}) = ([u]_X,[v]_Y)$, so f is surjective.

To wrap it all together, we have shown f to be a bijective linear transformation, so f is an isomorphism, and $(U\times V)/(X\times Y) \cong (U/X)\times(V/Y)$.

7. Let $C^\infty(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of infinitely-differentiable real-valued functions $f:\mathbb{R}\to\mathbb{R}$.

   (a) Let U denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which vanish at -2 and at π. That is,

   $$U = \left\{ f\in C^\infty(\mathbb{R}) \;\middle|\; f(-2)=0 \text{ and } f(\pi)=0 \right\}.$$

   (You do not need to prove that U is a subspace.) Prove that the quotient vector space $C^\infty(\mathbb{R})/U$ is finite-dimensional. What is its dimension?

   (Note: $C^\infty(\mathbb{R})$ is **very** infinite-dimensional!)

   Define a map $T: C^\infty(\mathbb{R}) \to \mathbb{R}^2$ by $T(f) = (f(-2), f(\pi))$. T is a linear transformation, because

   $$\begin{aligned} T(f+g) &= ((f+g)(-2),(f+g)(\pi)) = (f(-2)+g(-2), f(\pi)+g(\pi)) \\ &= (f(-2),f(\pi)) + (g(-2),g(\pi)) = T(f)+T(g) \end{aligned}$$

   and

   $$T(cf) = ((cf)(-2),(cf)(\pi)) = (c\cdot f(-2), c\cdot f(\pi)) = c\cdot(f(-2),f(\pi)) = cT(f).$$

   Furthermore f is in $\ker(T)$ if and only if $f(-2) = f(\pi) = 0$, so $U = \ker T$.

   We claim that $\mathrm{Im}T = \mathbb{R}^2$. Given $(\alpha,\beta)\in\mathbb{R}^2$, $f(x) = (x+2)\frac{\beta}{\pi+2} - (x-\pi)\frac{\alpha}{2+\pi}$ is a C^∞ function such that $T(f) = (\alpha,\beta)$. By the first isomorphism theorem, $C^\infty(\mathbb{R})/U = C^\infty(\mathbb{R})/\ker T \cong \mathrm{Im}T = \mathbb{R}^2$, so $\dim(C^\infty(\mathbb{R})/U) = 2$.

(b) Let W denote the subspace of $C^\infty(\mathbb{R})$ consisting of those functions which "vanish to n^{th} order at 0":

$$W = \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; f(0) = 0,\ \frac{df}{dx}(0) = 0, \dots,\ \text{and } \frac{d^n f}{dx^n}(0) = 0 \right\}.$$

Prove that the quotient vector space $C^\infty(\mathbb{R})/W$ is finite-dimensional and find a basis.

Define a map $T : C^\infty(\mathbb{R}) \to \mathbb{R}^{n+1}$ by $T(f) = (f(0), \frac{df}{dx}(0), ..., \frac{d^n f}{dx^n}(0))$. Because $\frac{d}{dx}$ is linear, we can use a similar argument to part (a) to show T is linear. Furthermore, f is in ker(T) if and only if $\frac{d^k f}{dx^k}(0) = 0$ for $0 \le k \le n$, so $W = \ker(T)$.

We claim that $\text{Im}(T) = \mathbb{R}^{n+1}$. Given $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1}$,

$$f = \sum_{k=0}^{n} \frac{\alpha_k}{k!} x^k$$

is a C^∞ function such that $T(f) = (\alpha_0, \alpha_1, \dots, \alpha_n)$. By the first isomorphism theorem, $C^\infty(\mathbb{R})/W = C^\infty(\mathbb{R})/\ker(T) \cong \text{Im}(T) = \mathbb{R}^{n+1}$, so $C^\infty(\mathbb{R})/W$ is finite-dimensional.

Note $\left\{ \left[\frac{x^k}{k!}\right]_W : 0 \le k \le n \right\}$ is a basis of $C^\infty(\mathbb{R})/W$ because its image under the isomorphism $C^\infty(\mathbb{R})/W \cong \mathbb{R}^{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$.

(c) Let $O(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid f(-x) = -f(x)\}$ be the set of **odd** smooth functions and let $E(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid f(-x) = f(x)\}$ be the set of **even** smooth functions. Prove that $O(\mathbb{R})$ and $E(\mathbb{R})$ are complementary. Prove that $C^\infty(\mathbb{R})/O(\mathbb{R}) \cong E(\mathbb{R})$. (You might want to find a way to use the First Isomorphism Theorem.)

Define a map $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by $T(f(x)) = \frac{1}{2}(f(x) + f(-x))$. Because function addition is linear, so is T. A function f is in ker(T) if and only if $f(-x) = -f(x)$, so $\ker(T) = O(\mathbb{R})$. Letting $g(x) = \frac{1}{2}(f(x) + f(-x))$, we see $g(-x) = g(x)$, so $\text{Im}(T) \subseteq E(\mathbb{R})$. To show $E(\mathbb{R}) \subseteq \text{Im}(T)$, for an $f \in E(\mathbb{R})$, note

$$T(f(x)) = \frac{1}{2}(f(x) + f(-x)) = \frac{1}{2}(f(x) + f(x)) = f(x),$$

since $f(-x) = f(x)$. Thus $\text{Im}(T) = E(\mathbb{R})$. Then by the first isomorphism theorem,

$$C^\infty(\mathbb{R})/O(\mathbb{R}) = C^\infty(\mathbb{R})/\ker(T) \cong \text{Im}(T) = E(\mathbb{R}).$$

We claim $T^2 = T$. To see this, for $f \in C^\infty(\mathbb{R})$,

$$\begin{aligned} T^2(f) &= T(\frac{1}{2}f(x) + \frac{1}{2}f(-x)) = \frac{1}{2}T(f(x)) + \frac{1}{2}T(f(-x)) \\ &= \frac{1}{2} \cdot \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2} \cdot \frac{1}{2}(f(-x) + f(x)) = \frac{1}{2}(f(x) + f(-x)) = T(f). \end{aligned}$$

By problem 3(a), $C^\infty(\mathbb{R}) = \ker(T) \oplus \ker(T - \mathbb{1})$. As argued previously, $\ker(T) = O(\mathbb{R})$. Furthermore, $(T - \mathbb{1})(f) = \frac{1}{2}(f(-x) - f(x))$, so f is in $\ker(T - \mathbb{1})$ if and only if $f(-x) = f(x)$. Thus $\ker(T - \mathbb{1}) = E(\mathbb{R})$, and $E(\mathbb{R})$ and $O(\mathbb{R})$ are complementary in $C^\infty(\mathbb{R})$.

Extended Glossary.
As has been shown in class, linear transformations form a vector space. The linear transformations from a vector space to its underlying field are particularly useful, with many interesting symmetries.

Definition 1. Let V be a vector space over $\mathbb{F}$. The **dual vector space** of V is the set of linear transformations from V to $\mathbb{F}$ and is denoted by V^*.

In other words, $V^* = \mathscr{L}(V, \mathbb{F})$, which has been shown to be a vector space over $\mathbb{F}$. In what follows, we will show that when V is finite dimensional, $\dim V^* = \dim V$ by finding a basis of V^*.

Theorem 2. *Let* $\dim V = n$ *and let* $v_1, v_2, \ldots, v_n$ *be a basis of* V. *For* $1 \leq k \leq n$, *let* $v_k^* : V \to \mathbb{F}$ *be the linear transformation defined by* $v_k^*(v_i) = 1$ *if* $i = k$ *and* $= 0$ *if* $i \neq k$. *Then the* v_k^* *form a basis of* V^*.

Proof. We first show that the v_k^* are linearly independent. Suppose for some $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, $\alpha_1 v_1^* + \cdots + \alpha_n v_n^* = 0$. This means $(\alpha_1 v_1^* + \cdots + \alpha_n v_n^*)(v) = 0$ for all $v \in V$. In particular, we have $(\alpha_1 v_1^* + \cdots + \alpha_n v_n^*)(v_k) = 0$ for all k. But since

$$(\alpha_1 v_1^* + \cdots + \alpha_n v_n^*)(v_k) = \alpha_1 v_1^*(v_k) + \cdots + \alpha_n v_n^*(v_k) = \alpha_k,$$

$\alpha_k = 0$ for all k. Thus the v_k^* are linearly independent.
To show that the v_k^* span V^*, let $f \in \mathscr{L}(V, \mathbb{F})$. We need to show that f can be written as a linear combination of the v_k^*. Let $\alpha_k = f(v_k)$ and define $g = \alpha_1 v_1^* + \cdots + \alpha_n v_n^*$. We will show that $f = g$. For all k, we have

$$g(v_k) = (\alpha_1 v_1^* + \cdots + \alpha_n v_n^*)(v_k) = \alpha_1 v_1^*(v_k) + \cdots + \alpha_n v_n^*(v_k) = \alpha_k = f(v_k).$$

Since f and g agree on a basis of V, they are the same linear transformation.
We conclude that $\dim V^* = \dim V = n$ and the v_k^* form a basis of V^*. □

In the following result, we show that a linear transformation between vector spaces induces a linear transformation between their duals.

Theorem 3. *Let* $T : V \to W$ *be a linear transformation. Then the map* $T^* : W^* \to V^*$ *given by* $T^*(f) = f \circ T$ *is well defined and is also a linear transformation.*

Proof. To show that T^* is well defined, we need to show that $T^*(f) \in V^*$ for any $f \in W^*$, i.e. $T^*(f)$ is a linear transformation on V. Let $v, v_1, v_2 \in V$ and $\alpha \in \mathbb{F}$. We have

$$T^*(f)(v_1 + v_2) = f(T(v_1 + v_2)) = f(T(v_1) + T(v_2)) = f(T(v_1)) + f(T(v_2)) = T^*(f)(v_1) + T^*(f)(v_2)$$

$$T^*(f)(\alpha v) = f(T(\alpha v)) = f(\alpha T(v)) = \alpha(f(T)(v)) = \alpha T^*(f)(v).$$

Therefore, T^* is well defined.
To show that T^* is a linear transformation, let $f, f_1, f_2 \in W^*$ and let $\alpha \in \mathbb{F}$. Then we have

$$T^*(f_1 + f_2)(v) = (f_1 + f_2)(T(v)) = f_1(T(v)) + f_2(T(v)) = T^*(f_1)(v) + T^*(f_2)(v) = (T^*(f_1) + T^*(f_2))(v)$$

$$T^*(\alpha f)(v) = (\alpha f)(T(v)) = \alpha(f(T(v))) = \alpha T^*(f)(v)$$

for all $v \in V$, as desired. □