1. Prove that any matrix $A \in M_n(\mathbb{C})$ is similar to its transpose A^{tr}. (Warning: If you think you have a way of proving this without working with the Jordan canonical form, it's probably wrong.)

First we will show that $J_m(\lambda)$ is similar to its transpose. Let B_m be the matrix such that $(B_m)_{ij} = \begin{cases} 1 & \text{if } i+j = m+1 \\ 0 & \text{otherwise} \end{cases}$. Note that B_m represents the permutation which reverses a vector, so $B_m^{-1} = B_m$. Then

$$(B_m J_m(\lambda))_{ij} = \sum_{k=1}^{m} (B_m)_{ik} J_m(\lambda)_{kj} = J_m(\lambda)_{m+1-i,j} = \begin{cases} \lambda & \text{if } i+j = m+1 \\ 1 & \text{if } i+j = m \\ 0 & \text{otherwise} \end{cases},$$

so

$$(B_m J_m(\lambda) B_m)_{ij} = \sum_{k=1}^{m} (B_m J_m(\lambda))_{ik} (B_m)_{kj} = (B_m J_m(\lambda))_{i,m+1-j} = \begin{cases} \lambda & \text{if } i = j \\ 1 & \text{if } i+1 = j \\ 0 & \text{otherwise} \end{cases}.$$

Thus $J_m(\lambda)^{tr} = B_m J_m(\lambda) B_m^{-1}$, as desired.

Let J be in Jordan canonical form and B be the block matrix:

$$J = \begin{bmatrix} J_{m_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{m_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{m_r}(\lambda_r) \end{bmatrix}, \qquad B = \begin{bmatrix} B_{m_1} & 0 & \cdots & 0 \\ 0 & B_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{m_r} \end{bmatrix}.$$

Then

$$B^2 = \begin{bmatrix} B_{m_1} & 0 & \cdots & 0 \\ 0 & B_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{m_r} \end{bmatrix} \cdot \begin{bmatrix} B_{m_1} & 0 & \cdots & 0 \\ 0 & B_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{m_r} \end{bmatrix} = \begin{bmatrix} I_{m_1} & 0 & \cdots & 0 \\ 0 & I_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{m_r} \end{bmatrix} = I,$$

so $B^{-1} = B$ and

$$BJB = \begin{bmatrix} B_{m_1} & 0 & \cdots & 0 \\ 0 & B_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{m_r} \end{bmatrix} \cdot \begin{bmatrix} J_{m_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{m_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{m_r}(\lambda_r) \end{bmatrix} \cdot \begin{bmatrix} B_{m_1} & 0 & \cdots & 0 \\ 0 & B_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{m_r} \end{bmatrix}$$

$$= \begin{bmatrix} B_{m_1}J_{m_1}(\lambda_1)B_{m_1} & 0 & \cdots & 0 \\ 0 & B_{m_2}J_{m_2}(\lambda_2)B_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{m_r}J_{m_r}(\lambda_r)B_{m_r} \end{bmatrix}$$

$$= \begin{bmatrix} J_{m_1}(\lambda_1)^{tr} & 0 & \cdots & 0 \\ 0 & J_{m_2}(\lambda_2)^{tr} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{m_r}(\lambda_r)^{tr} \end{bmatrix} = J^{tr}.$$

Thus J is similar to its transpose.

Any matrix $A \in M_n(\mathbb{C})$ is similar to its Jordan canonical form, so $A = PJP^{-1}$ for some invertible matrix P. Then

$$A^{tr} = (PJP^{-1})^{tr} = (P^{-1})^{tr}J^{tr}P^{tr} = (P^{tr})^{-1}J^{tr}P^{tr},$$

so A^{tr} is similar to J^{tr}. Similarity is an equivalence relation, and we have shown A^{tr} is similar to J^{tr}, which is similar to J, which is similar to A. Thus A is similar to its transpose.

2. Let $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\langle (\alpha, \beta), (\gamma, \delta) \rangle = 5\alpha \cdot \gamma - 2(\alpha \cdot \delta + \beta \cdot \gamma) + \beta \cdot \delta.$$

(a) Show that $\langle \cdot, \cdot \rangle$ is an inner product.

First we show it is symmetric. Note that for any $(\alpha, \beta), (\gamma, \delta) \in \mathbb{R}^2$, we have

$$\langle (\gamma, \delta), (\alpha, \beta) \rangle = 5\gamma \cdot \alpha - 2(\gamma \cdot \beta + \delta \cdot \alpha) + \delta \cdot \beta = 5\alpha \cdot \gamma - 2(\alpha \cdot \delta + \beta \cdot \gamma) + \beta \cdot \delta = \langle (\alpha, \beta), (\gamma, \delta) \rangle.$$

Now we show it is bilinear. For any $(\alpha, \beta), (\gamma, \delta), (\epsilon, \eta) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} \langle (\alpha, \beta) + \lambda(\gamma, \delta), (\epsilon, \eta) \rangle &= \langle (\alpha + \lambda\gamma, \beta + \lambda\delta), (\epsilon, \eta) \rangle \\ &= 5(\alpha + \lambda\gamma)\epsilon - 2[(\alpha + \lambda\gamma)\eta + (\beta + \lambda\delta)\epsilon] + (\beta + \lambda\delta)\eta \\ &= 5\alpha\epsilon + 5\lambda\gamma\epsilon - 2\alpha\eta - 2\lambda\gamma\eta - 2\beta\epsilon - 2\lambda\delta\epsilon + \beta\eta + \lambda\delta\eta \\ &= [5\alpha\epsilon - 2(\alpha\eta + \beta\epsilon) + \beta\eta] + \lambda[5\gamma\epsilon - 2(\gamma\eta + \delta\epsilon) + \delta\eta] \\ &= \langle (\alpha, \beta), (\epsilon, \eta) \rangle + \lambda \langle (\gamma, \delta), (\epsilon, \eta) \rangle. \end{aligned}$$

Finally we show it is non-degenerate. For any $(\alpha, \beta) \in \mathbb{R}^2$, we have

$$\langle (\alpha, \beta), (\alpha, \beta) \rangle = 5\alpha^2 - 4\alpha\beta + \beta^2 = \alpha^2 + (2\alpha - \beta)^2 \geq 0,$$

and

$$0 = \langle (\alpha, \beta), (\alpha, \beta) \rangle = \alpha^2 + (2\alpha - \beta)^2$$

if and only if $\alpha = 0$ and $2\alpha = \beta$ if and only if $(\alpha, \beta) = (0, 0)$. Thus $\langle \cdot, \cdot \rangle$ is an inner product.

(b) Determine the lengths of the vectors

$$(1,0), \quad (0,1), \quad (1,3), \quad (-1,2)$$

using this inner product.

First we compute a general formula:

$$\|(\alpha,\beta)\| = \sqrt{\langle(\alpha,\beta),(\alpha,\beta)\rangle} = \sqrt{\alpha^2 + (2\alpha-\beta)^2}.$$

Then

$$\|(1,0)\| = \sqrt{\langle(1,0),(1,0)\rangle} = \sqrt{1+2^2} = \sqrt{5},$$
$$\|(0,1)\| = \sqrt{\langle(0,1),(0,1)\rangle} = \sqrt{0+(-1)^2} = \sqrt{1} = 1,$$
$$\|(1,3)\| = \sqrt{\langle(1,3),(1,3)\rangle} = \sqrt{1+(2-3)^2} = \sqrt{2},$$
$$\|(-1,2)\| = \sqrt{\langle(-1,2),(-1,2)\rangle} = \sqrt{(-1)^2+(-2-2)^2} = \sqrt{17}.$$

(c) Draw a picture of the circle of radius 1 about the origin (i.e. the vectors of length one) in **this inner product**.

We want the unit circle about the origin in this inner product, so we want to draw the elements of

$$\{(a,b) : \|(a,b)\| = 1\} = \{(a,b) : \langle(a,b),(a,b)\rangle = 1\} = \left\{(a,b) : 5a^2 - 4ab + b^2 = 1\right\}$$

which is the equation of a rotated ellipse centred at the origin, with angle of rotation ϕ. First note a general ellipse is defined by the points (x,y) that satisfy the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B^2 - 4AC < 0$.

Now if we have an xy-coordinate system that is rotated by angle θ to form an $\tilde{x}\tilde{y}$ coordinate system, then a point (x,y) in the first coordinate system will become $\tilde{x}\tilde{y}$ in the second, where $x = \tilde{x}\cos\theta - \tilde{y}\sin\theta$ and $y = \tilde{x}\sin\theta + \tilde{y}\cos\theta$ (and $\tilde{x} = x\cos\theta + y\sin\theta$ and $\tilde{y} = -x\sin\theta + y\cos\theta$). To see why this is true, note a point (x,y) becomes $(r\cos\alpha, r\sin\alpha)$ in polar coordinates, so rotation by θ gives $(r\cos(\alpha+\theta), r\sin(\alpha+\theta))$. Then using trigonometric identities, we have $(r(\cos\alpha\cos\theta - \sin\alpha\sin\theta), r(\sin\alpha\cos\theta + \cos\alpha\sin\theta))$, which becomes $(r\cos\alpha\cos\theta - r\sin\alpha\sin\theta, r\sin\alpha\cos\theta + r\cos\alpha\sin\theta)$ which we can write as $(x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta)$.

Now to determine the angle of rotation, we introduce the change of variables $x = \tilde{x}\cos\theta - \tilde{y}\sin\theta$ and $y = \tilde{x}\sin\theta + \tilde{y}\cos\theta$ into $5x^2 - 4xy + y^2 = 1$ and note we want to eliminate the xy-term in the general equation (to generate an ellipse in standard position in the new system of coordinates), so we want to rotate the coordinate axes by angle $\theta \in [0,\pi]$, where $\cot(2\theta) = \frac{5-1}{-4} = -1$, so $\theta = 3\pi/8$ (this is because when introducing the change of variables in the equation above, the coefficient for $\tilde{x}\tilde{y}$ is $-10\cos\theta\sin\theta - 4(\cos^2\theta - \sin^2\theta) + 2\sin\theta\cos\theta$, which if we want to equal 0 generates precisely $\cot(2\theta) = \frac{5-1}{-4} = -1$).

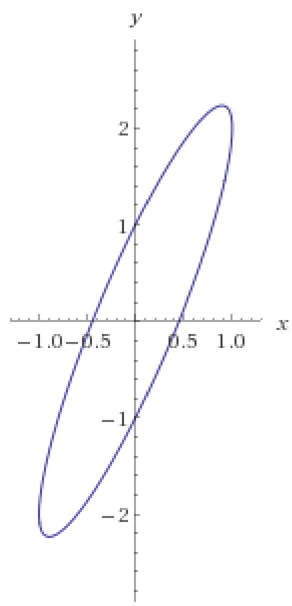
Now using trigonometric identities, we know $\cos 3\pi/4 = -\sqrt{2}/2$, and using the half angle formula, we get $\cos(3\pi/8) = \frac{\sqrt{2-\sqrt{2}}}{2}$ while $\sin(3\pi/8) = \frac{\sqrt{2+\sqrt{2}}}{2}$.

Then the equation $5x^2 - 4xy + y^2 = 1$ becomes

$$5\left(\frac{\sqrt{2-\sqrt{2}}}{2}\tilde{x} - \frac{\sqrt{2+\sqrt{2}}}{2}\tilde{y}\right)^2 - 4\left(\frac{\sqrt{2-\sqrt{2}}}{2}\tilde{x} - \frac{\sqrt{2+\sqrt{2}}}{2}\tilde{y}\right)\left(\frac{\sqrt{2+\sqrt{2}}}{2}\tilde{x} - \frac{\sqrt{2-\sqrt{2}}}{2}\tilde{y}\right) + \left(\frac{\sqrt{2+\sqrt{2}}}{2}\tilde{x} - \frac{\sqrt{2-\sqrt{2}}}{2}\tilde{y}\right)^2 = 1,$$

which reduces to $(3-2\sqrt{2})\tilde{x}^2 + (3+2\sqrt{2})\tilde{y}^2 = 1$, so $\frac{\tilde{x}^2}{\frac{1}{(3-2\sqrt{2})}} + \frac{\tilde{y}^2}{\frac{1}{(3+2\sqrt{2})}} = 1$.

This is the graph of a standard ellipse with respect to $\tilde{x}\tilde{y}$ coordinate system, where $a^2 = \frac{1}{(3-2\sqrt{2})}$ and $b^2 = \frac{1}{(3+2\sqrt{2})}$, which is rotated in the xy coordinate system by $3\pi/8$. We can represent the ellipse below (courtesy of Wolfram Alpha):



3. Let $V = C^\infty(\mathbb{R}, \mathbb{R})$ be the inner product space of continuous functions $\mathbb{R} \to \mathbb{R}$ with inner product

$$\langle f(t), g(t) \rangle = \int_0^1 f(t)g(t)dt.$$

Use the Gram-Schmidt process on the linearly independent set $\{t, e^t, \cos(\pi t)\}$ to get an orthogonal set. (You can use Wolfram Alpha or whatever else to calculate the integrals that come up when doing this. You can also in theory normalize the vectors, but that's an even worse mess than it already is!)

We use Wolfram Alpha to compute all integrals. Set $u_1 = t$. Then

$$u_2 = e^t - \frac{\langle e^t, u_1 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 = e^t - 3t.$$

Continuing, we have

$$\begin{aligned} u_3 &= \cos(\pi t) - \frac{\langle \cos(\pi t), u_2 \rangle}{\langle u_2, u_2 \rangle} \cdot u_2 - \frac{\langle \cos(\pi t, u_1 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 \\ &= \cos(\pi t) - \frac{\frac{6}{\pi^2} - \frac{1+e}{1+\pi^2}}{\frac{1}{2}(e^2 - 7)}(e^t - 3t) + \frac{6}{\pi^2} t. \end{aligned}$$

Since this doesn't simplify in any meaningful way, we leave our answers as they are.

4. Let $C^0([0,1], \mathbb{R})$ denote continuous $\mathbb{R}$-valued functions on the interval $[0,1]$. As above, this has inner product,

$$\langle f(t), g(t) \rangle = \int_0^1 f(t) g(t)\ dt.$$

This has a three-dimensional subspace $W = \mathcal{S}\text{pan}(\mathbb{1}, t, t^2)$. Compute the orthogonal projection of the following functions onto that subspace.

For both we will use Theorem 5.13. This is equivalent to solving the system

$$\begin{bmatrix} \langle \mathbb{1}, \mathbb{1} \rangle & \langle \mathbb{1}, t \rangle & \langle \mathbb{1}, t^2 \rangle \\ \langle t, \mathbb{1} \rangle & \langle t, t \rangle & \langle t, t^2 \rangle \\ \langle t^2, \mathbb{1} \rangle & \langle t^2, t \rangle & \langle t^2, t^2 \rangle \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \langle u(t), \mathbb{1} \rangle \\ \langle u(t), t \rangle \\ \langle u(t), t^2 \rangle \end{bmatrix}.$$

Letting $v = (c_1, c_2, c_3)$, $w = (\langle u(t), \mathbb{1} \rangle, \langle u(t), t \rangle, \langle u(t), t^2 \rangle)$, and

$$A = \begin{bmatrix} \langle \mathbb{1}, \mathbb{1} \rangle & \langle \mathbb{1}, t \rangle & \langle \mathbb{1}, t^2 \rangle \\ \langle t, \mathbb{1} \rangle & \langle t, t \rangle & \langle t, t^2 \rangle \\ \langle t^2, \mathbb{1} \rangle & \langle t^2, t \rangle & \langle t^2, t^2 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix},$$

we find that $v = A^{-1} w$, so we use Wolfram Alpha to compute

$$A^{-1} = 3 \begin{bmatrix} 3 & -12 & 10 \\ -12 & 64 & -60 \\ 10 & -60 & 60 \end{bmatrix}.$$

(a) $h(t) = t^3$

The orthogonal projection of h onto W is $c_1 \mathbb{1} + c_2 t + c_3 t^2$, where

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} \langle t^3, \mathbb{1} \rangle \\ \langle t^3, t \rangle \\ \langle t^3, t^2 \rangle \end{bmatrix} = 3 \begin{bmatrix} 3 & -12 & 10 \\ -12 & 64 & -60 \\ 10 & -60 & 60 \end{bmatrix} \cdot \begin{bmatrix} 1/4 \\ 1/5 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 1/20 \\ -3/5 \\ 3/2 \end{bmatrix}.$$

(b) $\ell(t) = \sqrt{t}$

The orthogonal projection of ℓ onto W is $c_1 \mathbb{1} + c_2 t + c_3 t^2$, where

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} \langle \sqrt{t}, \mathbb{1} \rangle \\ \langle \sqrt{t}, t \rangle \\ \langle \sqrt{t}, t^2 \rangle \end{bmatrix} = 3 \begin{bmatrix} 3 & -12 & 10 \\ -12 & 64 & -60 \\ 10 & -60 & 60 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ 2/5 \\ 2/7 \end{bmatrix} = \begin{bmatrix} 6/35 \\ 48/35 \\ -4/7 \end{bmatrix}.$$

These projections are the best approximations to the original functions that are in the subspace W. "Best" means closest in terms of the norm defined by this inner product.

5. Let V be an inner product space. A linear transformation $T : V \to V$ is called an **orthogonal reflection** if there exists a $n-1$-dimensional subspace W such that $T(w) = w$ for $w \in W$, and there exists a vector v orthogonal to W such that $T(v) = -v$. Show that given this v, and assuming v is scaled so $\|v\| = 1$, then T must be given by the formula

$$T(u) = u - 2\langle u, v\rangle v.$$

Note that since $\dim W = n-1$ and $\dim V = n$, we have $\dim W^\perp = 1$. Because v is in $W^\perp$ and $v \neq 0$, v spans $W^\perp$. Therefore it suffices to show that $T(w) = w - 2\langle w, v\rangle v$ for $w \in W$ and $T(v) = v - 2\langle v, v\rangle v$, since this means $T(u) = u - 2\langle u, v\rangle v$ on W and $W^\perp$. Because $\langle v, v\rangle = \|v\|^2 = 1$, we have

$$v - 2\langle v, v\rangle v = v - 2v = -v = T(v).$$

Furthermore, because v is orthogonal to W, $\langle w, v\rangle = 0$ for all $w \in W$, so

$$w - 2\langle w, v\rangle v = w - 0 = w = T(w),$$

which completes our proof.

6. Let $V = \mathcal{M}_{2\times 2}(\mathbb{C})$ be equipped with the inner product $\langle A, B\rangle = \mathrm{trace}(A^{\mathrm{tr}} \cdot \overline{B})$. Let $T : V \to \mathbb{C}$ be the map

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\right) = a_{11} - a_{22}.$$

Find a vector $W \in \mathcal{M}_{2\times 2}(\mathbb{C})$ so that $T(A) = \langle A, W\rangle$.

Let $W = \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}$. Then for any $A \in V$, we have

$$\langle A, W\rangle = \mathrm{trace}\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}^{\mathrm{tr}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\right) = \mathrm{trace}\begin{bmatrix} a_{11} & -a_{21} \\ a_{12} & -a_{22}\end{bmatrix} = a_{11} - a_{22} = T(A),$$

as desired.

7. Let $S, T : V \to V$ be linear operators on an inner product space, and let S^* and T^* denote their adjoints. Prove:

(a) $(S + \alpha \cdot T)^* = S^* + \overline{\alpha} \cdot T^*$.
Let $v, w \in V$. First note

$$\langle v, \alpha w\rangle = \overline{\langle \alpha w, v\rangle} = \overline{\alpha\langle w, v\rangle} = \overline{\alpha} \cdot \overline{\langle w, v\rangle} = \overline{\alpha}\langle v, w\rangle.$$

We have also $\langle T(v), w\rangle = \langle v, T^*(w)\rangle$ and $\langle S(v), w\rangle = \langle v, S^*(w)\rangle$. Then

$$\begin{aligned}\langle v, (S + \alpha \cdot T)^*(w)\rangle &= \langle (S + \alpha \cdot T)(v), w\rangle = \langle S(v) + \alpha \cdot T(v), w\rangle = \langle S(v), w\rangle + \langle \alpha \cdot T(v), w\rangle \\ &= \langle S(v), w\rangle + \alpha \cdot \langle T(v), w\rangle = \langle v, S^*(w)\rangle + \alpha \cdot \langle v, T^*(w)\rangle \\ &= \langle v, S^*(w)\rangle + \langle v, \overline{\alpha} \cdot T^*(w)\rangle = \langle v, (S^* + \overline{\alpha} \cdot T^*)(w)\rangle.\end{aligned}$$

Therefore $(S + \alpha \cdot T)^* = S^* + \overline{\alpha} \cdot T^*$.

(b) $(S \circ T)^* = T^* \circ S^*$.
Let $v, w \in V$. Then

$$\begin{aligned}\langle v, (S \circ T)^*(w)\rangle &= \langle (S \circ T)(v), w\rangle = \langle S(T(v)), w\rangle = \langle T(v), S^*(w)\rangle \\ &= \langle v, T^*(S^*(w))\rangle = \langle v, (T^* \circ S^*)(w)\rangle.\end{aligned}$$

Therefore $(S \circ T)^* = T^* \circ S^*$.

(c) $T + T^*$ is self-adjoint.
Let $v, w \in V$. Then

$$\begin{aligned}\langle v, (T + T^*)^*(w)\rangle &= \langle (T + T^*)(v), w\rangle = \langle T(v) + T^*(v), w\rangle = \langle T(v), w\rangle + \langle T^*(v), w\rangle \\ &= \langle T(v), w\rangle + \langle w, T^*(v)\rangle = \langle T(v), w\rangle + \langle T(w), v\rangle \\ &= \langle v, T^*(w)\rangle + \langle v, T(w)\rangle = \langle v, (T + T^*)(w)\rangle.\end{aligned}$$

Therefore $(T + T^*)^* = T + T^*$.

8. Play around with Wolfram Alpha! Compute the following:

(a) Diagonalize $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

We have $A = PDP^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & -1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$.

(b) Compute the exact values of the singular values of $B = \begin{bmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 3 & -1 & 0 & 1 \end{bmatrix}$.

Compute the SVD.

Note $B \cdot B^T = \begin{bmatrix} 14 & -5 \\ -5 & 11 \end{bmatrix}$. Then

$$0 = \det(BB^T - \lambda I) = (14 - \lambda)(11 - \lambda) - 25 = \lambda^2 - 25\lambda + 129$$

implies $\lambda = \frac{25 \pm \sqrt{109}}{2}$. Then the singular values are $\sqrt{\frac{25 \pm \sqrt{109}}{2}}$. For the SVD see Wolfram Alpha:

Input:

singular value decomposition	$\begin{pmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 3 & -1 & 0 & 1 \end{pmatrix}$

Result:

$M = U.\Sigma.V^*$

where

$$M = \begin{pmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 3 & -1 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} -0.802293 & 0.596931 \\ 0.596931 & 0.802293 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 4.20953 & 0 & 0 & 0 & 0 \\ 0 & 2.69812 & 0 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} -0.19059 & 0.22124 & 0.943456 & 0 & -0.156904 \\ 0.425414 & 0.892058 & -0.104828 & 0 & 0.110756 \\ -0.522984 & 0.145126 & 0 & 0 & 0.839897 \\ 0 & 0 & 0 & 1 & 0 \\ 0.713573 & -0.366366 & 0.314485 & 0 & 0.50763 \end{pmatrix}$$

(c) Compute the SVD of $C = \begin{bmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 3 & -1 & 0 & 1 \\ -1 & 1 & -2 & 0 & 0 \end{bmatrix}$.

Notice that C is just B with one extra row. Play around a bit with the extra row you might add and see if you see any patterns or nice examples. Write a little bit about what you found (if anything interesting).

Wolfram Alpha gives:

Input:

singular value decomposition	$\begin{pmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 3 & -1 & 0 & 1 \\ -1 & 1 & -2 & 0 & 0 \end{pmatrix}$

Result:

$M = U.\Sigma.V^{\dagger}$

where

$$M = \begin{pmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 3 & -1 & 0 & 1 \\ -1 & 1 & -2 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} -0.707107 & 0.673887 & 0.214186 \\ 0.565685 & 0.720853 & -0.400463 \\ 0.424264 & 0.162008 & 0.890928 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 4.58258 & 0 & 0 & 0 & 0 \\ 0 & 2.72938 & 0 & 0 & 0 \\ 0 & 0 & 1.59703 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} -0.246885 & 0.187544 & -0.42375 & -0.851064 & 0 \\ 0.46291 & 0.851685 & -0.194399 & 0.150188 & 0 \\ -0.617213 & 0.11098 & -0.596745 & 0.500626 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.586353 & -0.476596 & -0.653101 & 0.0500626 & 0 \end{pmatrix}$$

One thing we can note is that adding a third row to B that is in the linear span of the first two creates a matrix with only two singular values.

(d) Compute the Jordan decomposition of

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & 3 & -1 & 0 \end{bmatrix} \text{ and } E = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Wolfram Alpha gives:

Input:

Jordan decomposition	$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & 3 & -1 & 0 \end{pmatrix}$

Result:

$M = S.J.S^{-1}$

where

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & 3 & -1 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 0.4 - 0.8\,i & 0 & 0.4 + 0.8\,i \\ 0 & 0.8 + 0.4\,i & 0 & 0.8 - 0.4\,i \\ i & 1. + 2.\,i & -i & 1. - 2.\,i \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$J = \begin{pmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1.11022 \times 10^{-16} - 0.75\,i & 5.55112 \times 10^{-17} + 1.\,i & 3.66374 \times 10^{-17} - 0.5\,i & 0.5 - 3.66374 \times 10^{-17}\,i \\ 0.25 + 0.5\,i & 0.5 - 0.25\,i & -5.55112 \times 10^{-17} + 0.\,i & 5.55112 \times 10^{-17}\,i \\ -1.11022 \times 10^{-16} + 0.75\,i & -5.55112 \times 10^{-17} - 1.\,i & -3.66374 \times 10^{-17} + 0.5\,i & 0.5 + 3.66374 \times 10^{-17}\,i \\ 0.25 - 0.5\,i & 0.5 + 0.25\,i & 0.\,i & 0.\,i \end{pmatrix}$$

Input:

Jordan decomposition	$\begin{pmatrix} 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Result:

$M = S.J.S^{-1}$

where

$$M = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} -0.86676 & 1.86676 & 0.5 - 0.606658\,i & 0.5 + 0.606658\,i \\ 1.33107 & 0.286961 & -0.309017 + 1.58825\,i & -0.309017 - 1.58825\,i \\ -1.15372 & 0.535687 & 0.809017 + 0.981593\,i & 0.809017 - 0.981593\,i \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$J = \begin{pmatrix} -0.86676 & 0 & 0 & 0 \\ 0 & 1.86676 & 0 & 0 \\ 0 & 0 & 0.5 - 0.606658\,i & 0 \\ 0 & 0 & 0 & 0.5 + 0.606658\,i \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} -0.122911 + 3.18051 \times 10^{-17}\,i & 0.141805 - 6.5522 \times 10^{-18}\,i & -0.305409 - 5.55112 \times 10^{-17}\,i & 0.352356 - 2.77556 \times 10^{-17}\,i \\ 0.570125 - 7.28866 \times 10^{-18}\,i & 0.305409 - 7.85459 \times 10^{-18}\,i & -0.141805 + 6.93889 \times 10^{-18}\,i & -0.0759632 + 1.21431 \times 10^{-17}\,i \\ -0.223607 + 0.0435059\,i & -0.223607 - 0.184294\,i & 0.223607 - 0.184294\,i & 0.361803 + 0.070394\,i \\ -0.223607 - 0.0435059\,i & -0.223607 + 0.184294\,i & 0.223607 + 0.184294\,i & 0.361803 - 0.070394\,i \end{pmatrix}$$

Extended Glossary.
A symplectic form is a different sort of "geometric structure" we can impose on a vector space from an inner product. Symplectic forms are important in differential geometry and are the formalism behind the Hamiltonian formulation of classical mechanics!

Definition 1. Given a vector space V over a field $\mathbb{F}$, a *symplectic form* is a function $\omega : V \times V \to \mathbb{F}$ which satisfies the following:

- *linear in the first argument*: for any vectors $u, v, w \in V$ and scalar $\alpha \in \mathbb{F}$,
  - $\omega(\alpha \cdot u, v) = \alpha \cdot \omega(u, v)$,
  - $\omega(u + v, w) = \omega(u, w) + \omega(v, w)$,
- *alternating*: for any vector $v \in V$, $\omega(v, v) = 0_{\mathbb{F}}$, and
- *nondegenerate*: for every nonzero vector $u \in V$, there is a vector $v \in V$ such that $\omega(u, v) \neq 0_{\mathbb{F}}$.

A vector space equipped with a symplectic form is a *symplectic vector space*.

Remark 2. The second condition implies *anti-symmetry*, i.e. $\omega(u, v) = -\omega(v, u)$ for all $u, v \in V$. Suppose $\omega(v, v) = 0_{\mathbb{F}}$ for all $v \in V$. Then by bilinearity,

$$0_{\mathbb{F}} = \omega(u + v, u + v) = \omega(u, u) + \omega(u, v) + \omega(v, u) + \omega(v, v) = \omega(u, v) + \omega(v, u),$$

so $\omega(u, v) = -\omega(v, u)$ for all $u, v \in V$.

Remark 3. With anti-symmetry, we can show that ω is also linear in the second argument, hence implying *bilinearity*, i.e. linear in both arguments. For any vectors $u, v, w \in V$ and scalar $\alpha \in \mathbb{F}$,

$$\omega(u, \alpha v) = -\omega(\alpha v, u) = -\alpha\omega(v, u) = \alpha\omega(u, v), \quad \text{and}$$
$$\omega(u, v + w) = -\omega(v + w, u) = -\omega(v, u) - \omega(w, u) = \omega(u, v) + \omega(u, w),$$

as desired.

Example 4. Let $V = \mathbb{F}^{2n}$, I_n denote the $n \times n$ identity matrix, and $A = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Finally let $\omega(u, v) = u^{tr} A v$. We will check each of the conditions.

- For $u, v, w \in V$ and $\alpha \in \mathbb{F}$, we have
$$\omega(\alpha u, v) = (\alpha u)^{tr} A v = \alpha(u^{tr}) A v = \alpha(u^{tr} A v) = \alpha \cdot \omega(u, v), \quad \text{and}$$
$$\omega(u + v, w) = (u + v)^{tr} A w = (u^{tr} + v^{tr}) A w = u^{tr} A w + v^{tr} A w = \omega(u, w) + \omega(v, w),$$
  which shows linearity in the first argument.

- For $v = (v_1, \ldots, v_{2n}) \in \mathbb{F}^{2n}$, we have
$$\omega(v, v) = v^{tr} \cdot \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \cdot v = \begin{bmatrix} -v_{n+1} & -v_{n+2} & \cdots & v_{2n} & v_1 & v_2 & \cdots & v_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{2n} \end{bmatrix}$$
$$= \sum_{i=1}^{n} (-v_{n+i}) v_i + \sum_{j=1}^{n} v_j v_{n+j} = 0_{\mathbb{F}}.$$
  which shows alternation.

- Take any nonzero vector v. Then letting $v = (v_1, \dots, v_{2n})$, it must be that $v_k \neq 0_{\mathbb{F}}$ for some $1 \leq k \leq 2n$. If $k \leq n$, let $u = e_{n+k}$, and if $k > n$, let $u = e_{k-n}$, where e_i denotes the ith standard basis vector for $\mathbb{F}^{2n}$. Then

$$\omega(v, u) = v^{tr} \cdot \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \cdot u = \begin{bmatrix} -v_{n+1} & -v_{n+2} & \cdots & -v_{2n} & v_1 & v_2 & \cdots & v_n \end{bmatrix} \cdot u.$$

  If $k \leq n$, then $\omega(v, u) = v_k$, and if $k > n$, then $\omega(v, u) = -v_k$. In either case, $\omega(v, u) \neq 0_{\mathbb{F}}$, which shows nondegeneracy.

Example 5. For an inner product space, the inner product is *not* a symplectic form. An inner product $\langle \cdot, \cdot \rangle$ must satisfy $\langle v, v \rangle = 0$ if and only $v = 0$, which contradicts a symplectic form, where $\omega(v, v) = 0$ for all $v \in V$. For instance, if $V = C^0([0, 1], \mathbb{R})$ and $\omega(f, g) = \int_0^1 f(t)g(t)\, dt$ as in exercise 4, ω is not a symplectic form, as $\omega(1, 1) = 1 \neq 0_{\mathbb{R}}$.

Theorem 6. *If* V *is a finite-dimensional symplectic vector space, then* $\dim V$ *is even.*

Proof. We will prove this by induction on $\dim V$. First suppose $0 < \dim V \leq 2$, and take $v \neq 0_V$. Then there is a vector u such that $\omega(v, u) \neq 0_{\mathbb{F}}$. If u is a scalar multiple of v, then

$$\omega(v, u) = \omega(v, \alpha v) = \alpha\omega(v, v) = 0_{\mathbb{F}},$$

which contradicts our choice of u. Therefore $\{u, v\}$ is linearly independent, so $\dim V > 1$.
Now suppose that for every vector space W, if $\dim W \leq n - 1$ and W has a symplectic form, then $\dim W$ is even, and let V be a symplectic vector space of dimension n. Take a nonzero vector x. As before, there is a vector y such that $\omega(x, y) \neq 0_{\mathbb{F}}$ and $\{x, y\}$ is linearly independent. Let $X = \text{Span}(x, y)$ and define $X^\perp = \{v \in V : \omega(v, x) = \omega(v, y) = 0_{\mathbb{F}}\}$. Note that $X^\perp$ is a subspace because of the bilinearity of ω. We will show that $V = X \oplus X^\perp$ and ω is a symplectic form on $X^\perp$. Once we have shown these, since $\dim X^\perp < \dim V$ and $\dim X = 2$, we can conclude that $\dim X^\perp$ is even, so $\dim V = 2 + \dim X^\perp$ is also even.
Suppose $v \in X \cap X^\perp$. Since $v \in X$, there are scalars α, β such that $v = \alpha x + \beta y$. Then

$$0_{\mathbb{F}} = \omega(v, x) = \omega(\alpha x + \beta y, x) = \alpha\omega(x, x) + \beta\omega(y, x) = \beta\omega(y, x) = -\beta\omega(x, y),$$

$$0_{\mathbb{F}} = \omega(v, y) = \omega(\alpha x + \beta y, y) = \alpha\omega(x, y) + \beta\omega(y, y) = \alpha\omega(x, y),$$

which shows that $\alpha = \beta = 0_{\mathbb{F}}$, so $X \cap X^\perp = \{0\}_V$.
Now take any $v \in V$, and let $v' = \frac{\omega(v,x)}{\omega(y,x)}y + \frac{\omega(v,y)}{\omega(x,y)}x$. Note that $v' \in X$ and by bilinearity,

$$\omega(v', x) = \omega\left(\frac{\omega(v, x)}{\omega(y, x)}y + \frac{\omega(v, y)}{\omega(x, y)}x, x\right) = \frac{\omega(v, x)}{\omega(y, x)}\omega(y, x) + \frac{\omega(v, y)}{\omega(x, y)}\omega(x, x) = \omega(v, x)$$
$$\omega(v', y) = \omega\left(\frac{\omega(v, x)}{\omega(y, x)}y + \frac{\omega(v, y)}{\omega(x, y)}x, y\right) = \frac{\omega(v, x)}{\omega(y, x)}\omega(y, y) + \frac{\omega(v, y)}{\omega(x, y)}\omega(x, y) = \omega(v, y).$$

Then $\omega(v - v', x) = \omega(v - v', y) = 0_{\mathbb{F}}$, so $v - v'$ is in $X^\perp$, which shows that $V = X \oplus X^\perp$.
Finally we will show that ω is a symplectic form on $X^\perp$. Bilinearity and alternation still hold in $X^\perp$, so it remains to show that ω is nondegenerate on $X^\perp$. Take any nonzero $v \in X^\perp$. Since

ω is nondegenerate on V, there is a vector u in V such that $\omega(v, u) \neq 0_{\mathbb{F}}$. We can break down $u = u_{\parallel} + u_{\perp}$, where $u_{\parallel} \in X$ and $u_{\perp} \in X^{\perp}$. Then

$$\omega(v, u_{\perp}) = \omega(v, u - u_{\parallel}) = \omega(v, u) - \omega(v, u_{\parallel}) = 0_{\mathbb{F}},$$

because $\omega(v, w) = 0_{\mathbb{F}}$ for any $w \in X$. This shows that ω is nondegenerate in $X^{\perp}$, which completes our proof. □

Combining example 4 and theorem 6, we have the following:

Corollary 7. *A finite-dimensional vector space is symplectic if and only if its dimension is even.*