FRAMES and DEGENERATIONS of MONOMIAL RESOLUTIONS

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1. Introduction

We study the structure of (minimal) free resolutions of monomial ideals over a polynomial ring. This has been a very active area of research, and a number of new ideas and approaches were introduced in the last 8 years. In this paper, we introduce three new notions:

1) We introduce the frame of a free resolution. The frame is a complex of vector spaces which encodes the structure of the resolution entirely. A key idea in the paper is that the problem of constructing a monomial free resolution is essentially the problem of building its frame. There are three main sources of frames: homology complexes from algebraic topology (which yield cellular resolutions), dehomogenization of resolutions (see 2), and in some cases it is possible to construct frames directly (see Theorems 6.1 and 7.1).
2) Homogenization and dehomogenization of ideals are widely used (for example, to relate the defining ideals of affine and projective varieties). We introduce homogenization and dehomogenization of complexes.
3) We introduce degenerations of a monomial free resolution. Starting from a free resolution of a monomial ideal, a degeneration yields (under certain conditions) a free resolution of another monomial ideal.

Our constructions and results provide a framework which allows to treat several known constructions and results in [BPS,BS,GHP,GPW] as particular cases. This is explained in more detail in Remarks 3.3, 3.9, 3.11, 4.2, 4.7, 4.9.

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The results in Sections 5, 6, and 7 are applications of the techniques developed in Sections 3 and 4. On the one hand, the applications are interesting on their own, and on the other hand they provide an illustration of our methods.

In Section 5, we study Scarf complexes. Scarf simplicial complexes and resolutions were introduced in [BPS]. It is natural to ask what simplicial complexes appear as Scarf complexes of monomial ideals, and what simplicial complexes appear as Scarf complexes of Scarf ideals. In Theorem 5.3(1), we show that every finite simplicial complex, except the boundary of a simplex, is the Scarf complex of some monomial ideal. It was observed in [BPS, Lemma 2.1] that if a simplicial complex Δ is the Scarf complex of a Scarf ideal, then Δ is acyclic; furthermore, Example 5.2 in [BPS] shows that Δ need not be pure or shellable. In Theorem 5.3(2), we show that every finite acyclic simplicial complex is the Scarf complex of a Scarf ideal. As a consequence, we characterize the sequences of Betti numbers of Scarf ideals in Corollary 5.4.

In the proof of Theorem 5.3 we introduce nearly Scarf ideals. This is a class of monomial ideals with highly structured minimal free resolutions. The lcm-lattice of such a monomial ideal consists of the multidegrees of the faces of its Scarf complex and a top element (which is the lcm of all the minimal monomial generators of the ideal). In Theorem 6.1, we construct the minimal free resolution of any monomial ideal with such lcm-lattice. In Corollary 6.3 we list the numerical invariants of the minimal free resolution of a nearly Scarf ideal.

In Section 7, we obtain a lower bound on the Betti numbers of a monomial ideal in terms of its Scarf complex. The bound is sharp: it is attained by every nearly Scarf ideal. Furthermore, in Theorem 7.1(2), we describe the structure of the minimal free resolution of every monomial ideal with minimal Betti numbers among all monomial ideals with a fixed Scarf complex.

There are very few classes of monomial ideals (for example, Borel ideals [EK] and Scarf ideals [BPS]) for which the explicit minimal free resolution is known. Theorems 6.1 and 7.1 provide two new such classes: nearly Scarf ideals and ideals with minimal Betti numbers.

Resolutions supported by a simplicial complex were introduced in [BPS]. This was generalized to cellular resolutions, introduced in [BS] and studied in [BW]; such a resolution is supported by a regular cell complex. Furthermore, cellular resolutions were generalized in [JW] to be resolutions supported by CW-complexes. Nearly Scarf ideals provide many interesting examples of monomial ideals such that the minimal free resolution is supported by a CW-complex but not by a regular cell complex; such examples are presented in [Ve]. Nearly Scarf ideals also provide the first examples of monomial ideals whose minimal free resolutions do not admit any cellular structure, that is, are not supported by any CW-complex [Ve].

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2. Preliminaries

We introduce notation which will be used throughout the paper. Let M stand for a monomial ideal in the polynomial ring $S = k[x_1, \dots, x_n]$ minimally generated by monomials $m_1, \dots, m_r$.

2.1. GRADING.
The polynomial ring S is $\mathbf{N}^n$-graded by setting $\deg(x_i)$ to be the i'th standard vector in $\mathbf{N}^n$. Often we say that S is *multigraded* instead of $\mathbf{N}^n$-graded, and we say *multidegree* instead of $\mathbf{N}^n$-degree. For every $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{N}^n$ there exists a unique monomial of degree $\mathbf{a}$, namely $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. If an element g (say in a module) has $\mathbf{N}^n$-degree $\mathbf{a}$, then we say that is has *multidegree* $\mathbf{x}^{\mathbf{a}}$ and denote $\deg(g) = \mathbf{x}^{\mathbf{a}}$. Denote by $S(-\mathbf{x}^{\mathbf{a}})$ the free S-module generated by one element in multidegree $\mathbf{x}^{\mathbf{a}}$.

Every monomial ideal is multihomogeneous. Hence, there exists a minimal free resolution of S/M over S which is *multigraded* . Thus, we have *multigraded Betti numbers*

$$b_{i,m}(S/M) \;=\; \dim_k \operatorname{Tor}^S_{i,m}(S/M, k) \quad \text{for } i \geq 0,\ m \text{ a monomial.}$$

Therefore, the resolution can be written as

$$0 \longrightarrow \dots \longrightarrow \oplus_m\; S^{b_{3,m}}(-m) \longrightarrow \oplus_m\; S^{b_{2,m}}(-m) \longrightarrow \oplus_m\; S^{b_{1,m}}(-m) \longrightarrow S\,,$$

where the sum runs over all monomials m.

We denote by L_M the lattice with elements labeled by the least common multiples of $m_1, \dots, m_r$ ordered by divisibility. The atoms in L_M are $m_1, \dots, m_r$; the top element is $\operatorname{lcm}(m_1, \dots, m_r)$. The bottom element is 1 regarded as the lcm of the empty set. The least common multiple of elements in L_M is their join. Following [GPW] we call L_M the *lcm-lattice* of M. For $m \in L_M$ we denote by $(1, m)$ the open lower interval in L_M below m; it consists of all non-unit monomials in L_M that strictly divide m. The following result is proved in [GPW]:

2.2. THEOREM.
For $i \geq 1$ we have

$$b_{i,m}(S/M) \;=\; \begin{cases} \dim \tilde{\mathrm{H}}_{i-2}\big((1,m); k\big) & \text{if } m \in L_M \\ 0 & \text{if } m \notin L_M\;. \end{cases}$$

3. Homogenization and dehomogenization of complexes

3.1. Construction.

Let $\mathbf{U}$ be a complex of finite k-vector spaces with differential ∂ and a fixed basis, such that

- $\circ$ $U_i = 0$ for $i < 0$ and $i >> 0$,
- $\circ$ $U_0 = k$,
- $\circ$ $U_1 = k^r$,
- $\circ$ $\partial(w_j) = 1$ for each basis vector w_j in $U_1 = k^r$.

We call such a complex a *frame* (or an *r-frame*).

Let $\mathbf{G}$ be a multigraded complex of finitely generated free multigraded modules with differential d and a fixed multihomogeneous basis with multidegrees in L_M, such that

- $\circ$ $G_i = 0$ for $i < 0$ and $i >> 0$,
- $\circ$ $G_0 = S$,
- $\circ$ $G_1 = S(-m_1) \oplus \ldots \oplus S(-m_r)$,
- $\circ$ $d(w_j) = m_j$ for each basis element w_j of G_1.

We call such a complex an *M-complex*.

3.2. Construction.

Let $\mathbf{U}$ be an r-frame. We will construct by induction on the homological degree an M-complex $\mathbf{G}$ of free S-modules with differential d and call it the *M-homogenization* of $\mathbf{U}$.

Set $G_0 = S$ and $G_1 = S(-m_1) \oplus \ldots \oplus S(-m_r)$. Let $\bar{v}_1, \ldots, \bar{v}_p$ and $\bar{u}_1, \ldots, \bar{u}_q$ be the given bases of U_i and U_{i-1} respectively. Let $u_1, \ldots, u_q$ be the basis of $G_{i-1} = S^q$ chosen on the previous step of the induction. Introduce $v_1, \ldots, v_p$ that will be a basis of $G_i = S^p$. If $\partial(\bar{v}_j) = \sum_{1 \le s \le q} \alpha_{sj} \bar{u}_s$ with $\alpha_{sj} \in k$, then set

$$\deg(v_j) = \operatorname{lcm}\Big(\deg(u_s) \,|\, \alpha_{sj} \neq 0\Big), \text{ note that } \operatorname{lcm}(\emptyset) = 1$$

$$G_i = \oplus_{1 \le j \le p} \; S(-\deg(v_j))$$

$$d(v_j) = \sum_{1 \le s \le q} \alpha_{sj} \frac{\deg(v_j)}{\deg(u_s)} u_s \,.$$

We have that $\operatorname{Coker}(d_1) = S/M$. The differential d is multihomogeneous by construction. Straightforward verification shows that $\mathbf{G}$ is a complex. We say that the complex $\mathbf{G}$ is obtained from $\mathbf{U}$ by *M-homogenization*.

3.3. Remark.
Simplicial resolutions, introduced in [BPS], are obtained by M-homogenizing the augmented oriented chain complex of a simplicial complex. Cellular resolutions, introduced in [BS], are obtained by M-homogenizing the augmented oriented chain complex of a regular cell complex. CW-cellular resolutions, introduced in [JW], are obtained by M-homogenizing the complex computing the homology of a CW-complex.

3.4. Example.
Consider the monomial ideal $L = (x^5, xy, y^5)$ and the 3-frame

$$0 \longrightarrow k \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} k\,.$$

The L-homogenization of this frame is

$$0 \to A(-x^5y^5) \xrightarrow{\begin{pmatrix} y^4 \\ x^4 \\ 1 \end{pmatrix}} A(-x^5y) \oplus A(-xy^5) \oplus A(-x^5y^5) \xrightarrow{\begin{pmatrix} -y & 0 & y^5 \\ x^4 & -y^4 & 0 \\ 0 & x & -x^5 \end{pmatrix}}$$
$$A(-x^5) \oplus A(-xy) \oplus A(-y^5) \xrightarrow{\begin{pmatrix} x^5 & xy & y^5 \end{pmatrix}} A\,.$$

3.5. Construction.
Let $\mathbf{G}$ be an M-complex. We call

$$\mathbf{U} = \mathbf{G} \otimes S/(x_1 - 1, \ldots, x_n - 1)$$

the <u>*frame of* $\mathbf{G}$</u> (or the <u>*dehomogenization*</u> of $\mathbf{G}$). Thus, $\mathbf{U}$ is a finite complex of finite k-vector spaces with fixed basis and its differential matrices are obtained by setting $x_1 = 1, \ldots, x_n = 1$ in the differential matrices of $\mathbf{G}$. We say that the complex $\mathbf{U}$ is obtained from $\mathbf{G}$ by <u>*dehomogenization*</u>. Observe that:

3.6. Proposition.
If $\mathbf{G}$ *is the* M*-homogenization of a frame* $\mathbf{U}$*, then* $\mathbf{U}$ *is the frame of* $\mathbf{G}$.

3.7. Construction.
Let $\mathbf{G}$ be an M-complex, and let $m \in M$ be a monomial. Denote by $\mathbf{G}(\leq m)$ the subcomplex of $\mathbf{G}$ that is generated by the multihomogeneous basis elements of multidegrees dividing m.

Set $v = \operatorname{lcm}(m_i | m_i \text{ divides } m)$. Then $\mathbf{G}(\leq m) = \mathbf{G}(\leq v)$ because all the basis elements of $\mathbf{G}$ have multidegrees in L_M by Construction 3.2.

The following criterion for exactness is very useful:

3.8. THEOREM.
Let $\mathbf{G}$ be an M-complex.

(1) *For each monomial $m \in M$, the component of $\mathbf{G}$ of multidegree m is isomorphic to the frame of the complex $\mathbf{G}(\leq m)$.*
(2) *The complex $\mathbf{G}$ is a free multigraded resolution of S/M if and only if for all multidegrees $m \in L_M$ the frame of the complex $\mathbf{G}(\leq m)$ is exact.*

PROOF: Let P be a free multigraded module generated by t and suppose that P appears in homological degree i in $\mathbf{G}$. Then P contributes to the component of $\mathbf{G}$ of multidegree m if and only if the multidegree $\deg(t)$ divides m; in this case P contributes the one-dimensional vector space $\frac{m}{\deg(t)}\, k$. Therefore the component of $\mathbf{G}$ of multidegree m is isomorphic to the frame of the complex $\mathbf{G}(\leq m)$. We proved (1).

We will prove (2). The complex $\mathbf{G}$ is multigraded, so it suffices to check exactness in each multidegree. Note that $G_0/d(G_1) = S/M$. Therefore, it suffices to check exactness in each multidegree $m \in M$. By (1), it follows that the complex $\mathbf{G}$ is exact if and only if for all multidegrees $m \in M$ the frame of the complex $\mathbf{G}(\leq m)$ is exact.

Now, let $m \in M$ be a multidegree. Set $v = \operatorname{lcm}(m_i | m_i \text{ divides } m)$. Then $\mathbf{G}(\leq m) = \mathbf{G}(\leq v)$ because all the basis elements of $\mathbf{G}$ have multidegrees in L_M by Construction 3.2. Therefore, it suffices to consider only the multidegrees in L_M. □

3.9. REMARK.
Theorem 3.8 was proved in the following special cases:

- when the frame of $\mathbf{G}$ is the augmented oriented chain complex of a simplicial complex [BPS, Lemma 2.2].
- when the frame of $\mathbf{G}$ is the augmented oriented chain complex of a regular cell complex [BS, Proposition 1.2].

Our next result shows that a free resolution $\mathbf{F}_M$ of S/M contains as subcomplexes the minimal free resolutions for certain smaller monomial ideals.

3.10. THEOREM.
Let $u \in M$ be a monomial. Consider the monomial ideal $(M_{\leq u})$ generated by the monomials $\{m_i \,|\, m_i \text{ divides } u\}$. Fix a multihomogeneous basis of a multigraded free resolution $\mathbf{F}_M$ of S/M.

(1) *The subcomplex $\mathbf{F}_M(\leq u)$ is a free multigraded resolution of $S/(M_{\leq u})$.*

(2) If $\mathbf{F}_M$ is the minimal free multigraded resolution of S/M, then the subcomplex $\mathbf{F}_M(\leq u)$ is independent of the choice of basis.
(3) If $\mathbf{F}_M$ is minimal, then so is $\mathbf{F}_M(\leq u)$.

PROOF: First, note that replacing u by $\mathrm{lcm}(m_i | m_i$ divides $u)$ changes neither the ideal $(M_{\leq u})$ nor the complex $\mathbf{F}_M(\leq u)$. So, we can assume that $u \in L_M$.

By Theorem 3.8, it suffices to show that for every $m \in L_{(M_{\leq u})}$ the frame of the complex $\big(\mathbf{F}_M(\leq u)\big)(\leq m)$ is exact. The frame of $\big(\mathbf{F}_M(\leq u)\big)(\leq m)$ is equal to the frame of $\mathbf{F}_M(\leq u \wedge m)$, where $u \wedge m$ is the meet of u and m in the lcm-lattice L_M. Since $\mathbf{F}_M$ is exact, by Theorem 3.8 it follows that the frame of $\mathbf{F}_M(\leq u \wedge m)$ is exact. We proved (1).

(2) holds because the multidegrees of the basis elements in $\mathbf{F}_M$ are the same in any choice of basis; they are determined by the multigraded Betti numbers. $\square$

3.11. REMARK.
Theorem 3.10 was proved in the following special cases:
- when $\mathbf{F}_M$ is the minimal free multigraded resolution of S/M [GHP, Theorem 2.1].
- when $\mathbf{F}_M$ is a cellular resolution [BPS, Lemma 2.2].

3.12. EXAMPLE.
We illustrate Theorem 3.10. Let $A = k[x, y, z]$, $T = (x^2, xy, xz, y^2)$, and $m = xyz$. Then $(T_{\leq xyz}) = (xy, xz)$. The minimal free multigraded resolution of A/T is

$$\mathbf{F}_T: \quad 0 \to A \xrightarrow{\begin{pmatrix} z \\ x \\ -y \\ 0 \end{pmatrix}} A^4 \xrightarrow{\begin{pmatrix} y & 0 & z & 0 \\ -x & z & 0 & y \\ 0 & -y & -x & 0 \\ 0 & 0 & 0 & -x \end{pmatrix}} A^4 \xrightarrow{(x^2 \;\; xy \;\; xz \;\; y^2)} A \to 0\,.$$

The subcomplex $\mathbf{F}_T(\leq xyz)$ is

$$(\mathbf{F}_T)(\leq xyz): \quad 0 \to A \xrightarrow{\begin{pmatrix} z \\ -y \end{pmatrix}} A^2 \xrightarrow{(xy \;\; xz)} A \to 0\,.$$

It is the minimal free multigraded resolution of $A/(xy, xz)$.

4. Degeneration

4.1. DEFINITION.
Let M' be a monomial ideal in a polynomial ring S' over the same ground field k. We say that M is a <u>reduction</u> of M' if there exists a map $f : L_{M'} \to L_M$ which is a bijection

on the atoms and preserves lcm's. In what follows, we will use f to order the minimal monomial generators $m'_1, \dots, m'_r$ of M' so that $f(m'_i) = m_i$ for each i. We call the map f a *degeneration* . We say that M and M' are *lcm-equivalent* if f is an isomorphism.

4.2. Remark.

The following are special cases of degeneration:

- The ideal M is a reduction of its generic deformations, constructed in [BPS, Section 4].
- The radical $\mathrm{rad}(M)$ is a reduction of M. The degeneration map f maps each monomial to its radical.
- The ideal M and its polarization M_{pol} are lcm-equivalent.

4.3. Construction.

Let $\mathbf{F}'$ be a free multigraded resolution of S'/M' with a fixed multihomogeneous basis $u'_1, \dots, u'_p$. Let $f(\mathbf{F}')$ be the free S-module with basis denoted $f(u'_1), \dots, f(u'_p)$, so that for each i the element $f(u'_i)$ has the same homological degree as u'_i, and the multidegree of $f(u'_i)$ is $f(\deg(u'_i))$. We define differential ∂ on $f(\mathbf{F}')$ as follows: for a basis element u' if

$$\partial'(u') = \sum_{1 \le s \le q} \alpha_{sj} \frac{\deg(u')}{\deg(u'_s)}\, u'_s$$

with u'_s elements in the fixed basis, and $\alpha_{sj} \in k$, then we set

$$\partial\big(f(u')\big) = \sum_{1 \le s \le q} \alpha_{sj} \frac{f(\deg(u'))}{f(\deg(u'_s))}\, f(u'_s)\,.$$

We say that $f(\mathbf{F}')$ is an *f-degeneration* of $\mathbf{F}'$. Thus, $f(\mathbf{F}')$ is multigraded. Straightforward verification shows that $f(\mathbf{F}')$ is a complex. Note that $\mathbf{F}'$ and $f(\mathbf{F}')$ have the same frame.

The following important property holds by construction:

4.4. Lemma.

If w is an element in the fixed basis of $\mathbf{F}'$, then the corresponding basis element $f(w)$ in the complex $f(\mathbf{F}')$ has multidegree $f(\deg(w))$.

4.5. Example.

Let $A' = k[x,y]$, $N' = (x^5, x^2y^2, y^5)$ and $A = k[a,b,c]$, $N = (ab, bc, ac)$. Define $f : L_{N'} \to L_N$ by $f(x^5) = ab$, $f(x^2y^2) = bc$, $f(y^5) = ac$, $f(x^5y^2) = abc$, $f(x^2y^5) = abc$, $f(x^5y^5) = abc$. Then f is a bijection on the atoms and preserves lcm's; however f is not an isomorphism.

The minimal free multigraded resolution $\mathbf{F}_{N'}$ of A'/N' is:

$$\mathbf{F}_{N'}: \quad 0 \;\to\; A'^2 \xrightarrow{\begin{pmatrix} y^2 & 0 \\ -x^3 & y^2 \\ 0 & -x^3 \end{pmatrix}} A'^3 \xrightarrow{(x^5 \;\; x^2y^2 \;\; y^5)} A' \;\to\; 0\,.$$

The f-degeneration $f(\,\mathbf{F}_{N'}\,)$ is:

$$f(\,\mathbf{F}_{N'}\,): \quad 0 \;\to\; A^2 \xrightarrow{\begin{pmatrix} c & 0 \\ -a & a \\ 0 & -b \end{pmatrix}} A^3 \xrightarrow{(ab \;\; bc \;\; ac)} A \;\to\; 0\,.$$

4.6. THEOREM.
Let M be a reduction of M'. Let $\mathbf{F}'$ be a free multigraded resolution of S'/M' with a fixed multihomogeneous basis with degrees in $L_{M'}$. The f-degeneration $f(\mathbf{F}')$ is a free multigraded resolution of S/M.

PROOF: We apply Theorem 3.8. Clearly, $f(\mathbf{F}')$ is an M-complex.

For every monomial $m \in L_M$, consider the set of monomials $f^{-1}(m)$ and set

$$m' = \operatorname{lcm}\{\, v' \mid v' \in f^{-1}(m)\,\}\,.$$

Since f preserves lcm's, it follows that $f(m') = m$. Thus, m' is the top (greatest) element in $f^{-1}(m)$. By Lemma 4.4, it follows that $f(\mathbf{F}')(\leq m)$ and $\mathbf{F}'(\leq m')$ have the same frame. Since $\mathbf{F}'$ is exact, by Theorem 3.8 it follows that the frame of $f(\mathbf{F}')(\leq m)$ is exact. $\square$

4.7. REMARK.
Theorem 4.6 was proved in the following special cases:

- when $\mathbf{F}'$ is the minimal free multigraded resolution of S'/M' [GPW, Theorem 3.3].
- when M' is a generic deformation of M [BPS, Theorem 4.3].

The following result is useful in obtaining bounds for the Betti numbers:

4.8. THEOREM.
Let M be a reduction of M'. The total Betti numbers of S/M are smaller or equal to those of S'/M'.

PROOF: This follows from Theorem 4.6 applied to the minimal free multigraded resolution of S'/M'. $\square$

4.9. REMARK.
Theorem 4.8 was proved in the special case when M' is a generic deformation of M in [BPS,

Corollary 4.4]. We apply Theorem 4.8 in the next section, in order to obtain a lower bound on the Betti numbers in Corollary 7.1.

4.10. Construction.
Let M be a reduction of M'. Let $\mathbf{F}'$ be a multigraded free resolution of S'/M' with a fixed multihomogeneous basis. We say that the M-homogenization of the frame of $\mathbf{F}'$ is an *f-homogenization* of $\mathbf{F}'$, and denote it by $\tilde{f}(\mathbf{F}')$. The role of f in this construction is only to provide an ordering of the minimal monomial generators $m'_1, \ldots, m'_r$ of M' so that $f(m'_i) = m_i$ for each i.

4.11. Example.
This example illustrates that the f-degeneration and the f-homogenization could differ. Consider the monomial ideal $N' = (x^2, y^2, z^2)$ which lcm-equivalent to $N = (x^3, y^3, z^3)$ in the ring $A = k[x, y, z]$. Let $\mathbf{F}'$ be the non-minimal free resolution

$$\mathbf{F}': \quad 0 \to A^2 \xrightarrow{\begin{pmatrix} z^2 & -z^2 \\ x^2 & 0 \\ -y^2 & 0 \\ 0 & 1 \end{pmatrix}} A^4 \xrightarrow{\begin{pmatrix} y^2 & 0 & z^2 & y^2z^2 \\ -x^2 & z^2 & 0 & -x^2z^2 \\ 0 & -y^2 & -x^2 & 0 \end{pmatrix}} A^3 \xrightarrow{(x^2 \;\; y^2 \;\; z^2)} A \to 0\,.$$

Its frame is

$$0 \to k^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}} k^4 \xrightarrow{\begin{pmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}} k^3 \xrightarrow{(1 \;\; 1 \;\; 1)} k \to 0\,.$$

Therefore, the f-homogenization is

$$\tilde{f}(\mathbf{F}'): \quad 0 \to A^2 \xrightarrow{\begin{pmatrix} z^3 & -1 \\ x^3 & 0 \\ -y^3 & 0 \\ 0 & 1 \end{pmatrix}} A^4 \xrightarrow{\begin{pmatrix} y^3 & 0 & z^3 & y^3 \\ -x^3 & z^3 & 0 & -x^3 \\ 0 & -y^3 & -x^3 & 0 \end{pmatrix}} A^3 \xrightarrow{(x^3 \;\; y^3 \;\; z^3)} A \to 0\,.$$

On the other hand, the f-degeneration is

$$f(\mathbf{F}'): \quad 0 \to A^2 \xrightarrow{\begin{pmatrix} z^3 & -z^3 \\ x^3 & 0 \\ -y^3 & 0 \\ 0 & 1 \end{pmatrix}} A^4 \xrightarrow{\begin{pmatrix} y^3 & 0 & z^3 & y^3z^3 \\ -x^3 & z^3 & 0 & -x^3z^3 \\ 0 & -y^3 & -x^3 & 0 \end{pmatrix}} A^3 \xrightarrow{(x^3 \;\; y^3 \;\; z^3)} A \to 0\,.$$

From Constructions 4.3 and 4.10, we see that:

4.12. LEMMA.
Let M be a reduction of M'. Let $\mathbf{F}'$ be a multigraded free resolution of S'/M' with a fixed multihomogeneous basis with degrees in $L_{M'}$. The f-homogenization and the f-degeneration of $\mathbf{F}'$ coincide if and only if the following property (*) is satisfied: for every w in the fixed basis we have that $\deg(w) = \mathrm{lcm}\Big(\deg(u_s)\,|\,\alpha_{sj} \neq 0\Big)$, where $\partial(w) = \sum_{1\leq s\leq q} m_{sj}u_s$ with u_s elements in the fixed basis (note that each m_{sj} is a monomial multiplied by a scalar).

4.13. THEOREM.
Let M be a reduction of M'. Let $\mathbf{F}'$ be a minimal free multigraded resolution of S'/M' with a fixed multihomogeneous basis.

(1) The f-homogenization $\tilde{f}(\mathbf{F}')$ and the f-degeneration $f(\mathbf{F}')$ coincide and are the same free resolution of S/M.
(2) The free resolution $f(\mathbf{F}') = \tilde{f}(\mathbf{F}')$ does not depend on the choice of basis.
(3) If S/M and S'/M' have the same total Betti numbers, then the free resolution $f(\mathbf{F}') = \tilde{f}(\mathbf{F}')$ is minimal.
(4) If f is an isomorphism, then the free resolution $f(\mathbf{F}') = \tilde{f}(\mathbf{F}')$ is minimal.

PROOF: First, we will prove (1). We will show that Lemma 4.12 can be applied. Let w be a multihomogeneous element in some multihomogeneous basis of $\mathbf{F}'$ and

$$\partial'(w) = \sum_{1\leq s\leq q} a_{js}u_s\,,$$

where u_s are multihomogeneous basis elements; and note that each a_{sj} is a monomial multiplied by a scalar. Since $\mathbf{F}'$ is minimal, at least one of the coefficients a_{js} does not vanish. We will prove that $\deg(w) = \mathrm{lcm}(\deg(u_s)\,|\,a_{js} \neq 0\,)$. Assume the opposite. Therefore, there exists a monomial $b \neq 1$ such that

$$\partial'(w) = b\sum_{1\leq s\leq q} \tilde{a}_{js}u_s\,,$$

where $b\,\tilde{a}_{js} = a_{js}$. Since $\partial'^2(w) = 0$, it follows that $\partial'^2(\sum_{1\leq s\leq q}\tilde{a}_{js}u_s) = 0$. Therefore, $\partial'(w) \in b\,\mathrm{Ker}(\partial')$. By Nakayama's Lemma, it follows that $\partial'(w)$ is not an element in any multihomogeneous minimal system of generators of the kernel. On the other hand, $\partial'(w)$ is an element in a multihomogeneous minimal system of generators of the kernel because $\mathbf{F}'$ is minimal. This is a contradiction.

Thus, $\deg(w) = \operatorname{lcm}(\deg(u_s) \mid a_{js} \neq 0)$.

By Lemma 4.12, it follows that the f-homogenization $\tilde{f}(\mathbf{F}')$ and the f-degeneration $f(\mathbf{F}')$ coincide. By Theorem 4.6, $f(\mathbf{F}')$ is a free multigraded resolution of S/M.

(2) holds since the multidegrees of the basis elements in $\mathbf{F}'$ are the same in any choice of basis; they are determined by the multigraded Betti numbers. (3) is clear. Furthermore, (4) follows from (3) and Theorem 2.2. $\square$

4.14. Corollary.
The M-homogenization of any frame of the minimal multigraded free resolution $\mathbf{F}$ of S/M is $\mathbf{F}$.

4.15. Corollary.
Let M be a reduction of M'. Suppose that S/M and S'/M' have the same total Betti numbers. There exists a multihomogeneous basis of the minimal free multigraded resolution $\mathbf{F}$ of S/M, such that the M'-homogenization of the frame of $\mathbf{F}$ is a minimal free multigraded resolution of S'/M'.

Proof: Let $\mathbf{F}'$ be a minimal free multigraded resolution of S'/M' with a fixed multihomogeneous basis $\mathcal{F}'$. By Theorem 4.13, the M-homogenization of the frame of $\mathbf{F}'$ is $\mathbf{F}$ with a fixed multihomogeneous basis, which we denote by $\mathcal{F}$. Thus, in the bases $\mathcal{F}'$ and $\mathcal{F}$, the frames of $\mathbf{F}'$ and $\mathbf{F}$ coincide. Since the resolution $\mathbf{F}'$ is minimal, by Corollary 4.14 we have that the M'-homogenization of the frame of $\mathbf{F}'$ is $\mathbf{F}'$. Therefore, the M'-homogenization of the frame of $\mathbf{F}$ is $\mathbf{F}'$. $\square$

One may ask if Corollary 4.15 does not depend on the choice of basis, that is, if the M'-homogenization of any frame of $\mathbf{F}$ is a minimal multigraded free resolution of S'/M'. The answer is negative by Example 7.3.

4.16. Theorem.
Let M be a reduction of M'.

1. *If $\mathbf{F}'$ is a cellular free resolution of S'/M', then the f-homogenization and the f-degeneration of $\mathbf{F}'$ coincide.*
2. *Let $\mathbf{T}'$ be Taylor's resolution of S'/M' (with the basis used in the construction of Taylor's resolution). The f-degeneration and the f-homogenization of $\mathbf{T}'$ coincide, and are the Taylor's resolution of S/M.*
3. *Suppose that M and M' are generated by regular monomial sequences. Let $\mathbf{K}'$ be the Koszul resolution of S'/M' (with the basis used in the construction of the Koszul*

complex). The f-degeneration and the f-homogenization of $\mathbf{K}'$ coincide, and are the Koszul resolution of S/M.

Proof: (1) holds by Lemma 4.12.

(3) is a particular case of (2).

(2) holds because f preserves lcm's, and therefore we have that

$$f(\mathrm{lcm}(m'_i \,|\, i \in \sigma)) = \mathrm{lcm}\big(f(m'_i) \,|\, i \in \sigma\big)$$

for any subset σ of $\{1, \dots, r\}$. $\square$

5. Scarf complexes

5.1. Construction.

For each subset τ of $\{1, \dots, r\}$ we set $m_\tau = \mathrm{lcm}(m_i \mid i \in \tau)$; by convention, $\mathrm{lcm}(\emptyset) = 1$. The Scarf *complex* of M is the simplicial complex

$$\Omega_M = \Big\{\, \tau \subseteq \{1, \dots, r\} \mid \ m_\tau \neq m_\sigma \ \text{ for all } \ \sigma \subseteq \{1, \dots, r\} \ \text{ other than } \ \tau \,\Big\}\,.$$

In [BPS] it is shown that Ω_M coincides with a simplicial complex introduced by Scarf in the context of mathematical economics. Denote by $\mathbf{F}_{\Omega_M}$ the M-homogenization of the augmented oriented simplicial chain complex of Ω_M. Following [BPS], we call M a *Scarf ideal* if $\mathbf{F}_{\Omega_M}$ is the minimal free resolution of S/M, and we say that the complex $\mathbf{F}_{\Omega_M}$ is its *Scarf resolution* . In this case we say that Ω_M *supports* a Scarf resolution.

The multidegree of a vertex v_i in Ω_M is m_i. The multidegree of a face $\tau \in \Omega_M$ is $\deg(\tau) = \mathrm{lcm}\Big(\deg(v) \,|\, v \text{ is a vertex of } \tau\Big)$; this is the multidegree of the basis element τ in $\mathbf{F}_{\Omega_M}$. The multidegrees of the faces of Ω_M are called the *Scarf multidegrees* .

5.2. Example..

The Scarf complex of $L = (x^3, xy, y^5)$ has three vertices x^3, xy, y^5 and the two edges $\{x^3, xy\}$, $\{xy, y^5\}$.

5.3. Theorem.

(1) A finite simplicial complex with r vertices is the Scarf complex of a monomial ideal if and only if it is not the boundary of the simplex with r vertices.

(2) A finite simplicial complex Ω supports a Scarf resolution if and only if Ω is acyclic.

Proof: Let Ω be a finite simplicial complex. If Ω is a point or $\emptyset$, then (1) and (2) hold. Assume that Ω has at least two vertices.

(1) For each face τ of Ω introduce a variable x_τ. Consider the polynomial ring $B = k[x_\tau \mid \tau \in \Omega,\, \tau \neq \emptyset]$. Set the multidegree of each vertex v of Ω to be $\deg(v) = \prod_{v \notin \tau \in \Omega} x_\tau$. It follows that a face σ has multidegree

$$\deg(\sigma) = \operatorname{lcm}(\deg(v) \mid v \in \sigma) = \prod_{\sigma \not\subseteq \tau \in \Omega} x_\tau\,.$$

Therefore, every two faces have distinct multidegrees. Let Θ be the simplex on the vertices of Ω. If μ is a face of Θ and $\mu \notin \Omega$, then μ has multidegree z, where z is the product of all the variables. Let J_Ω be the ideal generated by the multidegrees of the vertices. The complex Ω has at least two nonfaces if and only if it is not the boundary of Θ. Therefore, Ω is the Scarf complex of the ideal J_Ω if and only if Ω is not the boundary of Θ.

(2) If Ω supports a Scarf resolution, then it is acyclic by Theorem 3.8 applied to the multidegree m that is the lcm of all the minimal monomial generators of the ideal. Now, suppose that Ω is acyclic. We will show that the ideal J_Ω constructed in (1) is a Scarf ideal. The lcm of its minimal monomial generators is z. The lcm-lattice consists of Scarf multidegrees (including the bottom element 1) and the top element z. The interval $[1, z)$ is the face poset of Ω. Therefore, the order complex of the open interval $(1, z)$ is the barycentric subdivision of Ω (which is homotopic to Ω), so it is acyclic. By Theorem 2.2 it follows that all nonzero Betti numbers of S/J_Ω are concentrated in the multidegrees of the faces of Ω. By Lemma 3.1 in [BPS], we conclude that Ω supports the Scarf resolution of J_Ω. □

As a corollary, we characterize the possible sequences of Betti numbers of Scarf ideals:

5.4. Corollary.
Let $b_0 = 1, b_1, b_2, \ldots$ be a finite sequence of natural numbers. For each $i \geq 0$, set $\alpha_i = \sum_{j \geq 0} (-1)^j b_{i+j}$. The sequence $b_1, b_2, \ldots$ is the sequence of total Betti numbers of a Scarf ideal if and only the sequence $\alpha_0, \alpha_1, \ldots$ is the Hilbert function of the quotient $k[x_1, \ldots, x_{\alpha_1}]/\big(T + (x_1^2, \ldots, x_{\alpha_1}^2)\big)$ for some squarefree monomial ideal T.

Proof: By Theorem 5.3, it follows that $b_1, b_2, \ldots$ is the sequence of Betti numbers of a Scarf ideal if and only if $b_0 = 1, b_1, b_2, \ldots$ is the f-vector of an acyclic complex Ω, that is, b_i is the number of i-dimensional faces of Ω. The f-vectors of acyclic simplicial complexes are characterized in [Ka]. □

6. Nearly Scarf ideals

Throughout this section (except in 6.5), Ω is a a finite simplicial complex with at least 2 vertices. We say that the ideal J_Ω, constructed in the proof of Theorem 5.3(1), is the nearly-Scarf ideal of Ω. It is a squarefree ideal in the polynomial ring $B = k[x_\tau \,|\, \tau \in \Omega\,, \tau \neq \emptyset]$. If Ω is acyclic, then J_Ω is a Scarf ideal. We use the notation introduced in the previous section.

6.1. THEOREM.
Let J be a monomial ideal in S whose lcm-lattice consists of the Scarf multidegrees (including the bottom element 1) and a top element y. Let Ω be the Scarf complex of J, and

$$\mathbf{C}: \quad 0 \to C_{\dim(\Omega)}(\Omega, k) \to \ldots \to C_0(\Omega, k) \to C_{-1}(\Omega, k) \to 0$$

be the oriented augmented homology chain complex of Ω with differential ∂. For each i, choose a set $\{q_1, \ldots, q_p\}$ of cycles whose classes in $\mathrm{Ker}(\partial_i)/\mathrm{Im}(\partial_{i+1})$ form a basis and set

$$\phi_i: \; k^{\dim(\tilde{\mathrm{H}}_i(\Omega,k))} \to \mathrm{Ker}(\partial_i)$$

$$e_j \mapsto q_j\,,$$

where $e_1, \ldots$ are the standard basis elements. Let $\mathbf{U}$ be the complex

$$\mathbf{U}: \quad 0 \to k^{\dim \tilde{\mathrm{H}}_{\dim(\Omega)}(\Omega,k)} \to \ldots \to C_i(\Omega, k) \oplus k^{\dim \tilde{\mathrm{H}}_{i-1}(\Omega,k)} \to \ldots \to C_{-1}(\Omega, k) \to 0$$

with differential $\partial \oplus \phi$. The J-homogenization of the complex $\mathbf{U}$ is the multigraded minimal free resolution of S/J.

PROOF: Let $\sigma_1 + \ldots + \sigma_p$, where $\sigma_j \in \Omega$, be a cycle that is non-trivial in the homology $\tilde{\mathrm{H}}_i(\Omega, k))$. We will prove that $\mathrm{lcm}(\deg(\sigma_j) \,|\, 1 \leq j \leq p) = y$. Assume the opposite. It follows that $\mathrm{lcm}(\deg(\sigma_j) \,|\, 1 \leq j \leq p) = \deg(\tau)$ for some $\tau \in \Omega$. By the definition of the Scarf complex, it follows that all the faces $\sigma_1, \ldots, \sigma_p$ are subfaces of τ. But τ is a simplex, contradicting the fact that $\sigma_1 + \ldots + \sigma_p$ is non-trivial in homology. Thus, $\mathrm{lcm}(\deg(\sigma_j) \,|\, 1 \leq j \leq p) = y$.

Denote by $\mathbf{G}$ the J-homogenization of the complex $\mathbf{U}$. We will apply Theorem 3.8 in order to show that $\mathbf{G}$ is exact. Let $m \in L_J$. First, suppose that $m = y$. Then the frame of $\mathbf{G}(\leq m)$ is the complex $\mathbf{U}$, which is exact. Now, suppose that $m = \deg(\tau)$ for some $\tau \in \Omega$. Then the frame of $\mathbf{G}(\leq m)$ is the oriented augmented homology chain complex of the simplex τ, so it is exact. □

6.2. COROLLARY.
The ideals J and J_Ω have isomorphic lcm-lattices. The J_Ω-homogenization of the complex $\mathbf{U}$ (in Theorem 6.1) is the multigraded minimal free resolution of B/J_Ω.

PROOF: Set z to be the product of all the variables in the ring $B = k[x_\tau \,|\, \tau \in \Omega, \tau \neq \emptyset]$. Recall that the lcm-lattice of J_Ω consists of the Scarf multidegrees (including the bottom element 1) and the top element z. $\square$

We denote by $|\Omega|$ the number of nonempty faces of the complex Ω; it is equal to the degree of the monomial z. For a face $\sigma \in \Omega$, the degree of the monomial $\deg(\sigma) = \prod_{\{\tau\in\Omega|\,\sigma\not\subseteq\tau\}} x_\tau$ is equal to the number $|\{\tau \in \Omega|\, \sigma \not\subseteq \tau\}|$. Furthermore, let

$$\tilde{\chi}(\Omega, k) = \sum_{i=-1}^{\dim(\Omega)} (-1)^i \dim \tilde{\mathrm{H}}_i(\Omega, k)$$

be the reduced Euler characteristic of Ω.

6.3. Corollary.
The Hilbert series of B/J_Ω is

$$\frac{\tilde{\chi}(\Omega, k)\, t^{|\Omega|} \;-\; \sum_{\sigma\in\Omega} (-1)^{\dim(\sigma)}\, t^{|\{\tau\in\Omega|\,\sigma\not\subseteq\tau\}|}}{(1-t)^{|\Omega|}}$$

Furthermore,

$$\mathrm{pd}(B/J_\Omega) = \begin{cases} \dim(\Omega)+1 & \textit{if}\;\; \tilde{\mathrm{H}}_{\dim(\Omega)}(\Omega,k) = 0 \\ \dim(\Omega)+2 & \textit{otherwise,} \end{cases}$$

$$\mathrm{codim}(J_\Omega) = 2$$

$$\mathrm{reg}(J_\Omega) = \max\Big\{\; |\Omega| - \min\{i \mid \tilde{\mathrm{H}}_{i-2}(\Omega,k) \neq 0\}\;, \;\max\{|\{\tau\in\Omega|\,\sigma\not\subseteq\tau\}| - \dim(\sigma) - 1 \mid \sigma\in\Omega\,\}\Big\}.$$

Proof: The formulas for the Hilbert series and the projective dimension follow from Corollary 6.2. The formula for the regularity holds because

$$\begin{aligned}\mathrm{reg}(J_\Omega) = \max\Big\{ & \max\{\,|\Omega| - p \,\big|\, b_{p,z}(S/J_\Omega) \neq 0\}\,, \\ & \max\{\,(\text{the degree of } \deg(\sigma)\,) - p \,\big|\, b_{p,\deg(\sigma)}(S/J_\Omega) \neq 0,\ \sigma\in\Omega\,\}\Big\} \\ = \max\Big\{ & \max\{\,|\Omega| - i \,\big|\, \tilde{\mathrm{H}}_{i-2}(\Omega,k) \neq 0\,\}\;, \\ & \max\{|\{\tau\in\Omega|\,\sigma\not\subseteq\tau\}| - \dim(\sigma) - 1 \,\big|\, \sigma\in\Omega\,\}\Big\}.\end{aligned}$$

It remains to compute the codimension. The ideal J_Ω is squarefree, so it is the Stanley-Reisner ideal of a simplicial complex Δ on vertex set $\{\emptyset \neq \tau \in \Omega\}$. We have that $\dim(B/J_\Omega) = \dim(\Delta) + 1$. The ring B has $|\Omega|$ variables, so $\operatorname{codim}(J_\Omega) = |\Omega| - \dim(\Delta) - 1$.

By the definition of J_Ω it follows that $\{\tau_1, \ldots, \tau_i\} \in \Delta$ if and only if for each $\emptyset \neq \tau \in \Omega$ there exists a $\sigma \in \Omega$ such that $\sigma \not\supseteq \tau$ and $\sigma \notin \{\tau_1, \ldots, \tau_i\}$. Therefore, Δ has no faces of dimension $|\Omega| - 2$. Also, for every two disjoint nonempty $\tau_1, \tau_2 \in \Omega$, we have that $\{\emptyset \neq \tau \in \Omega\} \setminus \{\tau_1, \tau_2\}$ is a face in Δ. Hence, $\dim(\Delta) = |\Omega| - 3$. □

6.4. COROLLARY.
The ring S/J_Ω is Cohen-Macaulay if and only if Ω is a forest.

PROOF: By the Auslander-Buchsbaum formula, S/J_Ω is Cohen-Macaulay if and only if $\operatorname{pd}(S/J_\Omega) = \operatorname{codim}(J_\Omega) = 2$. This happens if and only if either $\dim(\Omega) = 0$ and $\tilde{\mathrm{H}}_0(\Omega, k) \neq 0$, or $\dim(\Omega) = 1$ and $\tilde{\mathrm{H}}_1(\Omega, k) = 0$. In the former case Ω is a set of points, in the latter Ω is a collection of trees. □

6.5. REMARK.
As a consequence, we obtain a new proof of the following well-known fact in algebraic topology: for a finite simplicial complex Ω, one has

$$\sum_{i=-1}^{\dim(\Omega)} (-1)^i \dim \tilde{\mathrm{H}}_i(\Omega, k) = \sum_{\sigma \in \Omega} (-1)^{\dim(\sigma)} .$$

PROOF: Suppose that Ω has r vertices. If Ω is the boundary of the simplex with r vertices, or a point, or $\emptyset$, then the formula holds. Otherwise, consider the nearly Scarf ideal J_Ω. Let

$$h(t) = \tilde{\chi}(\Omega, k)\, t^{|\Omega|} - \sum_{\sigma \in \Omega} (-1)^{\dim(\sigma)}\, t^{|\{\tau \in \Omega \,|\, \sigma \not\subseteq \tau\}|}$$

be the numerator of the Hilbert series from Corollary 6.3. Since $\operatorname{codim}(J_\Omega) \neq 0$, it follows that $h(1) = 0$. Hence

$$0 = \tilde{\chi}(\Omega, k) - \sum_{\sigma \in \Omega} (-1)^{\dim(\sigma)} .$$

□

7. Monomial ideals with smallest Betti numbers

Using nearly-Scarf ideals, we obtain a lower bound for the Betti numbers of a monomial ideal. Furthermore, we describe the structure of the minimal free resolution for a monomial ideal with minimal Betti numbers among all monomial ideals with a fixed Scarf complex.

Let Ω be the Scarf complex of a monomial ideal M. For a chain q in Ω, we define $\deg(q)$ to be the lcm of the degrees of the faces in its support. For a monomial $m \in L_M$, we define the subcomplex $\Omega_{\preceq m} = \{\tau \in \Omega \,|\, \deg(\tau) \text{ divides } m\}$ of Ω.

7.1. THEOREM.
Let Ω be the Scarf complex of a monomial ideal M. Denote by $f_i(\Omega)$ the number of i-dimensional faces of Ω.

(1) We have that

$$b_i(S/M) \geq f_{i-1}(\Omega) + \dim \widetilde{\mathrm{H}}_{i-2}\big(\Omega; k\big)\,.$$

(2) Suppose that for each i,

$$b_i(S/M) = f_{i-1}(\Omega) + \dim \widetilde{\mathrm{H}}_{i-2}\big(\Omega; k\big)\,.$$

Then there exists a basis of $\widetilde{\mathrm{H}}_*\big(\Omega; k\big)$, which satisfies the following property:

$$\begin{array}{l} \textit{for every multidegree } m \in L_M, \\ \textit{the elements whose degrees divide } m \textit{ form a basis of } \widetilde{\mathrm{H}}_*\big(\Omega_{\preceq m}; k\big). \end{array} \tag{7.2}$$

Moreover, using any basis that satisfies (7.2) as the choice of cycles in Theorem 6.1, the M-homogenization of the frame $\mathbf{U}$ is the minimal free resolution of S/M.

PROOF: First, we will prove (1). We use the notation and the construction in the proof of Theorem 5.3(1). Let z be the product of all the variables in the polynomial ring $B = k[x_\tau \,|\, \tau \in \Omega,\, \tau \neq \emptyset]$. Consider the map $f : L_M \to L_{J_\Omega}$ that preserves each monomial that is the multidegree of a face in Ω, and maps all other monomials to z. This map is a bijection on the atoms, and preserves lcm's. Hence, the nearly-Scarf ideal J_Ω is a reduction of the ideal M. By Theorem 4.8

$$b_i(S/M) \geq b_i(B/J_\Omega)$$

for all i.

We will compute the Betti number $b_i(B/J_\Omega)$. By Theorem 2.2 we get

$$b_{i,z}(B/J_\Omega) = \dim \widetilde{\mathrm{H}}_{i-2}\big(\,(1,z)\,; k\big) = \dim \widetilde{\mathrm{H}}_{i-2}\big(\Omega; k\big)\,,$$

since the order complex of the open interval $(1,z)$ (in L_{J_Ω}) and Ω are homotopic. By Theorem 2.2 we also get

$$\sum_{u \in L_{J_\Omega},\, u \neq z} b_{i,u}(B/J_\Omega) = f_{i-1}(\Omega)\,.$$

Therefore,

$$
\begin{aligned}
b_i(B/J_\Omega) &= b_{i,z}(B/J_\Omega) + \sum_{u \in L_{J_\Omega},\, u \neq z} b_{i,u}(B/J_\Omega) \\
&= f_{i-1}(\Omega) + \dim \widetilde{\mathrm{H}}_{i-2}\big(\Omega; k\big)\,.
\end{aligned}
$$

Thus, (1) is proved.

We will prove (2). We use the notation introduced in Theorem 6.1. Denote by $\mathbf{F}_M$ the minimal free resolution of S/M over S.

First, we will show that there exists a basis with the desired properties. The M-homogenization $\mathbf{C}_M$ of the frame $\mathbf{C}$ is the algebraic Scarf complex of M. Fix a basis $\mathcal{F}$ of $\mathbf{F}_M$ that contains the basis $\mathcal{C}$ of $\mathbf{C}_M$. We can write $\mathcal{F} = \mathcal{C} \cup \mathcal{V}$ as a disjoint union. The number of elements in the set $\mathcal{F}_i$ is $b_i(S/M) = f_{i-1}(\Omega) + \dim \widetilde{\mathrm{H}}_{i-2}\big(\Omega; k\big)$. The number of elements in the set $\mathcal{C}_i$ is $f_{i-1}(\Omega)$. Therefore, the number of elements in the set $\mathcal{V}_i$ is $\dim \widetilde{\mathrm{H}}_{i-2}\big(\Omega; k\big)$.

Consider the frame $\mathbf{U}$ of $\mathbf{F}_M$; denote it's differential by d and it's restriction to the subcomplex $\mathbf{C}$ by ∂. Since $\mathbf{F}_M$ is a free resolution, by Theorem 3.8(2) it follows that $\mathbf{U}$ is exact.

We will show by induction on $i \geq 1$ that $d_i(\mathcal{V}_i) \subseteq \operatorname{span}(\mathcal{C}_{i-1})$ is a set of cycles whose images in $\widetilde{\mathrm{H}}_{i-2}\big(\Omega; k\big)$ form a basis. For $i = 1$ the statement is obvious since $d_1 = \partial_0$. Let $i > 1$. The induction hypothesis guarantees that $\operatorname{Ker}(d_{i-1}) = \operatorname{Ker}(\partial_{i-2})$ since for any $v \in \operatorname{span}(\mathcal{V}_{i-1})$ and $c \in \operatorname{span}(\mathcal{C}_{i-1})$ the equality $0 = d_{i-1}(v + c) = d_{i-1}(v) + \partial_{i-2}(c)$ implies that $v = 0$. Since $\mathbf{U}$ is exact, the vectors $d_i(\mathcal{V}_i) \cup d_i(\mathcal{C}_i)$ span $\operatorname{Ker}(d_{i-1}) = \operatorname{Ker}(\partial_{i-2})$, so the cycles $d_i(\mathcal{V}_i)$ generate $\operatorname{Ker}(\partial_{i-2})/\operatorname{Im}(\partial_{i-1})$. Since the number of elements in $\mathcal{V}_i$ is $\dim \widetilde{\mathrm{H}}_{i-2}(\Omega, k)$, it follows that their classes form a basis.

Thus, $\mathcal{F} = \mathcal{C} \cup \mathcal{V}$ and $\mathcal{V}$ is a basis of $\widetilde{\mathrm{H}}_*\big(\Omega; k\big)$. Let $m \in L_M$. By Theorem 3.8(2), the frame of $\mathbf{F}_M(\leq m)$ is exact. By Construction 3.7, the basis of this frame consists of the elements in $\mathcal{F}$ whose degrees divide m. Since $\mathbf{C}_M(\leq m)$ is the oriented augmented homology chain complex of $\Omega_{\preceq m}$, it follows that the elements in $\mathcal{V}$ whose degrees divide m form basis of $\widetilde{\mathrm{H}}_*\big(\Omega_{\preceq m}; k\big)$.

We have shown that every multihomogeneous basis of $\mathbf{F}_M$, which contains $\mathcal{C}$, satisfies condition (7.2).

Now, let $\mathcal{V}$ be a basis of $\widetilde{\mathrm{H}}_*\big(\Omega; k\big)$ that satisfies condition (7.2). We will show that the M-homogenization $\mathbf{G}$ of the frame $\mathbf{U}$ is the minimal free resolution of S/M. For every $m \in L_M$, the frame of $\mathbf{G}(\leq m)$ is

$$
\begin{aligned}
&0 \to k^{\dim \tilde{\mathrm{H}}_{\dim(\Omega_{\preceq m})}(\Omega_{\preceq m}, k)} \to \dots \\
&\dots \to C_i(\Omega_{\preceq m}, k) \oplus k^{\dim \tilde{\mathrm{H}}_{i-1}(\Omega_{\preceq m}, k)} \to \dots \to C_{-1}(\Omega_{\preceq m}, k) \to 0\,,
\end{aligned}
$$

so it is exact. By Theorem 3.8(2), it follows that **G** is exact. Therefore, **G** is a multigraded free resolution of S/M. Since the ranks of the free modules in **G** coincide with the Betti numbers, we conclude that the resolution is minimal. $\square$

7.2. Remark.
The weaker lower bound $b_i(S/M) \geq f_{i-1}(\Omega)$ is proved in [BPS, remark before Lemma 3.1].

7.3. Example.
This example shows that Corollary 4.15 and Theorem 7.1(2) depend on the choice of basis. We construct two ideals M and M' such that:

- ∘ M is a reduction of M'
- ∘ M is nearly Scarf
- ∘ M and M' have the same Scarf complex
- ∘ M and M' have the same total Betti numbers
- ∘ both M and M' satisfy equalities in Theorem 7.1(1),

but there exists a multihomogeneous basis of the minimal free resolution **F** of S/M so that the M'-homogenization of **F** is not exact, so is not a free resolution of S'/M'.

Let Ω be the simplicial complex on 7 vertices that consists of two empty triangles and an edge with a common vertex: denote by $a, \dots, f$ the vertices and

$$\begin{aligned} &j_1 = \{a,b\},\ j_2 = \{b,c\},\ j_3 = \{a,c\},\ j_4 = \{a,d\}, \\ &j_5 = \{d,e\},\ j_6 = \{a,e\},\ j_7 = \{a,f\} \end{aligned}$$

the edges. Let M be the nearly Scarf ideal J_Ω. Also, consider the ideal

$$M' = (\deg(a), \deg(b), \deg(c),\ x\deg(d), x\deg(e), \deg(f)\,)\,.$$

We consider both ideals in the polynomial ring $V = k[a, \dots, f, j_1, \dots, j_7, x]$. The monomial $z = a \dots f j_1 \dots j_7$ is the top element in L_M. The lcm-lattice $L_{M'}$ consists of L_M and the monomial xz. The two ideals have the same Scarf complex and the same total Betti numbers. However, $2 = b_3^V(V/M) = b_{3,z}^V(V/M)$ is in one multidegree, whereas $2 = b_3^V(V/M')$ is in two different multidegrees $b_{3,z}^V(V/M') = b_{3,xz}^V(V/M') = 1$.

Using the chains

$$q_1 = j_1 + j_2 - j_3 + j_4 + j_5 - j_6 \quad \text{and} \quad q_2 = j_4 + j_5 - j_6,$$

which form basis of $\widetilde{\mathrm{H}}_*(\Omega;k)$, we obtain the following complex $\mathbf{U}$ as in Theorem 6.1:

$$\mathbf{U}: \quad 0 \to k^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix}} k^7 \xrightarrow{\begin{pmatrix} -1 & 0 & -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}} k^6 \xrightarrow{(1\ 1\ 1\ 1\ 1\ 1)} k \to 0\,.$$

By Theorem 6.1, the M-homogenization of $\mathbf{U}$ is the minimal free resolution $\mathbf{F}$ of V/M. However, the M'-homogenization $\mathbf{G}$ of $\mathbf{U}$ is not a minimal free resolution of V/M' since it is not exact by Theorem 3.8(2): the frame of $\mathbf{G}(\leq z)$ is isomorphic to the reduced homology chain complex of the subcomplex of Ω supported on the vertices a, b, c, f, which is not acyclic.

On the other hand, using the chains

$$p_1 = j_1 + j_2 - j_3 + j_4 + j_5 - j_6 \quad \text{and} \quad p_2 = j_4 + j_5 - j_6,$$

which form basis of $\widetilde{\mathrm{H}}_*(\Omega;k)$, we obtain the following complex $\mathbf{U}'$ as in Theorem 6.1:

$$\mathbf{U}': \quad 0 \to k^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \\ 1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}} k^7 \xrightarrow{\begin{pmatrix} -1 & 0 & -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}} k^6 \xrightarrow{(1\ 1\ 1\ 1\ 1\ 1)} k \to 0\,.$$

The chains p_1 and p_2 satisfy condition (7.2) in Theorem 7.1(2). Hence $\mathbf{V}$ is a common frame for $\mathbf{F}$ and $\mathbf{F}'$. Thus, the M'-homogenization of $\mathbf{V}$ is the minimal free resolution of V/M'.

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