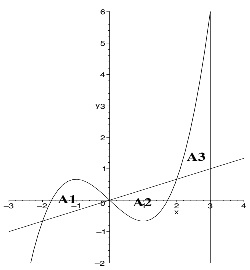
WEEK 1 HW SOLUTIONS

MATH 122



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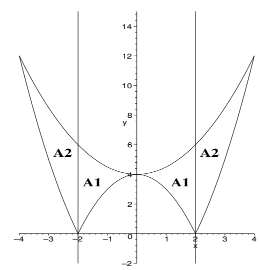

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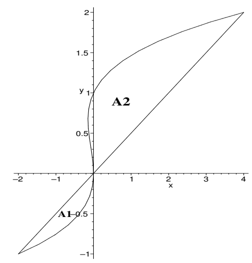

30

§5.1: Area Between Curves

12. The area is: $A = A1 + A2 + A3$
A1: $a = -2$ and $b = 0$: $f(x) - g(x) = (\frac{x^3}{3} - x) - \frac{x}{3} = \frac{x^3}{3} - \frac{4}{3}x = \frac{1}{3}(x^3 - 4x)$. Thus, $A1 = \frac{1}{3}\int_{-2}^{0}(x^3 - 4x)\,dx = \frac{1}{3}[\frac{x^4}{4} - 2x^2]_{-2}^{0} = 0 - \frac{1}{3}(4 - 8) = \frac{4}{3}$.
A2: $a = 0$, and we find b by solving for the intersection of $y = \frac{x^3}{3} - x$ and $y = \frac{x}{3}$: $\frac{x^3}{3} - x = \frac{x}{3} \Rightarrow \frac{x^3}{3} - \frac{4}{3}x = 0 \Rightarrow \frac{x}{3}(x-2)(x+2) = 0 \Rightarrow x = -2, 0,$ or 2. So, $b = 2$. Moreover, $f(x) - g(x) = \frac{x}{3} - (\frac{x^3}{3} - x) = -\frac{1}{3}(x^3 - 4x)$. Therefore, $A2 = -\frac{1}{3}\int_{0}^{2}(x^3 - 4x)\,dx = -\frac{1}{3}[\frac{x^4}{4} - 2x^2]_{0}^{2} = -\frac{1}{3}(4 - 8) = \frac{4}{3}$.
A3: $a = 2$ and $b = 3$ and, again, $f(x) - g(x) = \frac{1}{3}(x^3 - 4x)$. Thus, $A3 = \frac{1}{3}\int_{2}^{3}(x^3 - 4x)\,dx = \frac{1}{3}[(\frac{81}{4} - 18) - (\frac{16}{4} - 8)] = \frac{25}{12}$.
Therefore, the area is: $A = \frac{4}{3} + \frac{4}{3} + \frac{25}{12} = \frac{19}{4}$.

22. Since both of the functions in question are even, the graphs are symmetric about the y-axis. Thus, the area is $A = 2(A1 + A2)$. We first need to find the limits of integration. Treat $|x^2 - 4|$ as two functions. That is,

$$|x^2 - 4| = \begin{cases} x^2 - 4, & x \leq -2 \text{ or } x \geq 2 \\ 4 - x^2, & -2 < x < 2. \end{cases}$$

Then, for $x \geq 2$ or $x \leq -2$, $x^2 - 4 = \frac{x^2}{2} + 4 \Rightarrow \frac{x^2}{2} = 8 \Rightarrow x = \pm 4$. For $-2 < x < 2$, $4 - x^2 = \frac{x^2}{2} + 4 \Rightarrow -\frac{x^2}{2} = 0 \Rightarrow x = 0$. Thus, $A1 = \int_0^2 \frac{x^2}{2} + 4 - (4 - x^2)\,dx = \int_0^2 \frac{3x^2}{2}\,dx = [\frac{x^3}{2}]_0^2 = 4$, and $A2 = \int_2^4 \frac{x^2}{2} + 4 - (x^2 - 4)\,dx = \int_2^4 -\frac{x^2}{2} + 8\,dx = [-\frac{x^3}{6} + 8x]_2^4 = [-\frac{64}{6} + 32 + \frac{8}{6} - 16] = 16 - \frac{56}{6}$. Putting everything together, we see that the area is: $A = 2(4 + 16 - \frac{56}{6}) = \frac{64}{3}$.

30. First, we need to find the limits of integration. $y^3 - y^2 = 2y \Rightarrow y = 0$ or $y^2 - y - 2 = 0 \Rightarrow y = -1, 0,$ or 2. Then the area is: $A = A1 + A2$. $A1 = \int_{-1}^{0} y^3 - y^2 - 2y\,dy = [\frac{y^4}{4} - \frac{y^3}{3} - y^2]_{-1}^{0} = 0 - (\frac{1}{4} + \frac{1}{3} - 1) = \frac{5}{12}$. $A2 = \int_0^2 2y - (y^3 - y^2)\,dy = [y^2 - \frac{y^4}{4} + \frac{y^3}{3}]_0^2 = (4 - \frac{16}{4} + \frac{8}{3}) - 0 = \frac{8}{3}$. Thus, the area is: $A = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$.

§5.2: Finding Volume by Slicing

2. (a) $A = \pi(\text{radius})^2$ and the radius $= \sqrt{x}$, so $A = \pi(\sqrt{x})^2 = \pi x$.

4. The diameters are given as running from $y = x^2$ to $y = 2-x^2$, so the length of the diameter at x is $2-x^2-x^2 = 2(1-x^2)$. Thus, the radius at x is $(1-x^2)$. Hence the area of the cross-section at x is $A(x) = \pi(\text{radius})^2 = \pi(1-x^2)^2 = \pi(1-2x^2+x^4)$. We're given that x runs from -1 to 1, so the volume is: $V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \pi(1 - 2x^2 + x^4)\,dx = \pi[x - \frac{2x^3}{3} + \frac{x^5}{5}]_{-1}^{1} = 2\pi(1 - \frac{2}{3} + \frac{1}{5}) = \frac{16\pi}{15}$.

10. Since the triangles are isosceles, we know that the area of each triangle $A = \frac{1}{2}(\text{leg})(\text{leg})$. Since the planes that cut out the triangles run perpendicular to the y-axis, we need to express the area in terms of y. Solving $x^2 + y^2 = 1$ for x yields $x = \sqrt{1-y^2}$. Thus, the length of each leg is $\sqrt{1-y^2} - (-\sqrt{1-y^2}) = 2\sqrt{1-y^2}$. Thus, $A(y) = \frac{1}{2}(2\sqrt{1-y^2})^2 = 2(1-y^2)$. We're given the bounds, so the volume is: $V = \int_{-1}^{1} 2(1-y^2)\,dy = 2[y - \frac{y^3}{3}]_{-1}^{1} = \frac{8}{3}$.

§5.3: Volumes of Solids of Revolution – Disks and Washers

4. To find the limits of integration, we need to know when $R(x) = \sin(x)\cos(x)$ is zero. However, $R(x) = 0$ whenever $\sin(x) = 0$ or $\cos(x) = 0$, so this happens at $x = 0$ and next at $x = \frac{\pi}{2}$. Thus, $V = \int_0^{\pi/2} \pi(R(x))^2\,dx = \pi\int_0^{\pi/2} \sin^2(x)\cos^2(x)\,dx = \pi\int_0^{\pi/2} \frac{(\sin(2x))^2}{4}\,dx = \pi\int_0^{\pi} \frac{1}{8}\sin^2(u)\,du = \frac{\pi}{8}[\frac{u}{2} - \frac{1}{4}\sin(2u)]_0^{\pi} = \frac{\pi^2}{16}$. The two possibly mysterious steps are due to the double-angle formula for sine: $\sin(2x) = 2\sin(x)\cos(x)$ and a u-substitution with $u = 2x$.
   Note: as usual, there were MANY ways to do this integral – the above is just one example.

8. This one is much easier than the previous exercise. $R(x) = x - x^2$, and $R(x) = 0$ at $x = 0$ and $x = 1$. Thus, $V = \int_0^1 \pi(x-x^2)^2\,dx = \pi\int_0^1 x^2 - 2x^3 + x^4\,dx = \pi[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}]_0^1 = \pi(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}) = \frac{\pi}{30}$.

20. We're revolving around the y-axis, so we need everything in terms of y. Since $\arctan(0) = 0$ and $\arctan(1) = \frac{\pi}{4}$, we have our limits of integration (or you could just look at the graph in the book). The outer radius $R(y) = 1$ and the inner radius $r(y) = \tan(y)$. Thus, by the washer method the volume is: $V = \int_0^{\pi/4} \pi(R(y)^2 - r(y)^2)\,dy = \pi\int_0^{\pi/4} 1 - \tan^2(y)\,dy = \pi\int_0^{\pi/4} 2 - \sec^2(y)\,dy = \pi[2y - \tan(y)]_0^{\pi/4} = \frac{\pi^2}{2} - \pi$. (since, if we divide the World's Best Trig Identity® by $\cos^2(x)$, we find that $\tan^2(x) + 1 = \sec^2(x)$).

38. In both parts of this problem, our washers will need to run perpendicular to the y-axis, so we need everything in terms of y. Our limits of integration are 0 and 2, and the function is $x = \frac{y}{2}$

(a) In this case, the inner radius $r(y) = 0$ and the outer radius $R(y) = 1 - \frac{y}{2}$. So the washer method gives us that the volume is: $V = \int_0^2 \pi(1 - \frac{y}{2})^2\,dy = \pi \int_0^2 1 - y + \frac{y^2}{4}\,dy = \pi[y - \frac{y^2}{2} + \frac{y^3}{12}]_0^2 = \frac{2\pi}{3}$.
(b) Now, we have that $r(y) = 1$ and $R(y) = 2 - \frac{y}{2}$. This yields $V = \pi \int_0^2 (2 - \frac{y}{2})^2 - 1\,dy = \pi \int_0^2 3 - 2y + \frac{y^2}{4}\,dy = \pi[3y - y^2 + \frac{y^3}{12}]_0^2 = \frac{8\pi}{3}$.