HW 6 SOLUTIONS

MATH 122

§7.6: Improper Integrals

2.

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^{1.001}} &= \lim_{b\to\infty}\int_1^b \frac{dx}{x^{1.001}} \\
&= \lim_{b\to\infty}[-1000x^{-.001}]_1^b \\
&= \lim_{b\to\infty}(-1000)(b^{-.001}-1) = 1000.
\end{aligned}$$

4.

$$\begin{aligned}
\int_0^4 \frac{dx}{\sqrt{4-x}} &= \lim_{b\to 4^-}\int_0^b \frac{dx}{\sqrt{4-x}} \\
&= \lim_{b\to 4^-}\int_4^{4-b} -\frac{du}{\sqrt{u}} \\
&= \lim_{b\to 4^-}[-2u^{1/2}]_4^{4-b} \\
&= \lim_{b\to 4^-}[-2\sqrt{4-b}+2\sqrt{4}] = 4.
\end{aligned}$$

8.

$$\begin{aligned}
\int_0^1 \frac{dr}{r^{.999}} &= \lim_{b\to 0^+}\int_b^1 \frac{dr}{r^{.999}} \\
&= \lim_{b\to 0^+}[1000r^{.001}]_b^1 \\
&= \lim_{b\to 0^+} 1000[1-b^{.001}] = 1000.
\end{aligned}$$

16.

$$\begin{aligned}
\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds &= \lim_{b\to 2^-}\int_0^b \frac{s+1}{\sqrt{4-s^2}}\,ds \\
&= \lim_{b\to 2^-}\left(\int_4^{4-b^2} -\frac{1/2}{\sqrt{u}}\,du + \int_0^b \frac{ds}{2\sqrt{1-(s/2)^2}}\right) \\
&= \lim_{b\to 2^-}[-\sqrt{u}]_4^{4-b^2} + [\arcsin(s/2)]_0^b \\
&= \lim_{b\to 2^-}[-\sqrt{4-b^2}+2+\arcsin(b/2)-0] = 2+\frac{\pi}{2}.
\end{aligned}$$

32.

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \lim_{c\to 1^-}\int_0^c \frac{dx}{\sqrt{1-x}} + \lim_{d\to 1^+}\int_d^2 \frac{dx}{\sqrt{x-1}}$$
$$= \lim_{c\to 1^-}[-2\sqrt{1-x}]_0^c + \lim_{d\to 1^+}[2\sqrt{x-1}]_d^2$$
$$= \lim_{c\to 1^-}[-2\sqrt{1-c}+2] + \lim_{d\to 1^+}[2-2\sqrt{d-1}] = 4.$$

64. Remember that we're just proving convergence (or divergence); we don't really care what the value of the integral is. Since the function we're integrating is even, $\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}} = \int_0^{\infty}\frac{2dx}{e^x+e^{-x}}$. Moreover, since $e^x+e^{-x}\geq e^x$ we see that $\frac{1}{e^x+e^{-x}} \leq \frac{1}{e^x}$. So, by the direct comparison test, $\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}} = \int_0^{\infty}\frac{2dx}{e^x+e^{-x}} \leq \int_0^{\infty}\frac{2dx}{e^x} = 2$. Therefore, the integral in question converges.

§8.1: Sequences

32. The sequence is non-decreasing. Here's a proof:
$1 \leq 2(2n+5)$ for all $n>0$, thus $a_n \leq a_n\cdot 2(2n+5) = a_n \cdot \frac{2(n+2)(2n+5)}{n+2} = \frac{(2n+3)!(2n+4)(2n+5)}{(n+1)!(n+2)} = \frac{(2n+5)!}{(n+2)!} = \frac{(2(n+1)+3)!}{((n+1)+1)!} = a_{n+1}$. Notice how I STARTED with an obvious truth and concluded what I was trying to prove. This is the opposite direction in which you would work out the proof on your scratchpaper, but it is how you should write up your final draft of any proof.
The sequence is not bounded above. Here's a proof using the technique called "proof by contradiction"; we assume something and reach a contradiction, thus proving that we made a false assumption:
Assume the sequence IS bounded above. If that is so, then there is a large positive number M such that $M \geq \frac{(2n+3)!}{(n+1)!}$ for all natural numbers n. Thus, $M(n+1)! \geq (2n+3)! = (n+1)!(n+2)\cdots(n+(n-1))(2n)(2n+1)(2n+2)(2n+3)$ for all n. This leads us to conclude that $M \geq (n+2)\cdots(n+(n-1))(2n)(2n+1)(2n+2)(2n+3)$ for all n. However, this is ridiculous, since M would have to be bigger than any n. No such "biggest number" exists, so we have a contradiction. Therefore, our assumption (that the sequence is bounded above) is false.

34. The sequence is non-decreasing:
Since $\frac{1}{n} > \frac{1}{n+1}$, we have that $-\frac{2}{n} < -\frac{2}{n+1}$. This gives us that $-\frac{2}{n}+\frac{2}{n+1}$ is negative. Also, $\frac{1}{2^n} > \frac{1}{2^{n+1}}$, so $\frac{1}{2^n}-\frac{1}{2^{n+1}}$ is positive. So, clearly, $-\frac{2}{n}+\frac{2}{n+1} \leq \frac{1}{2^n}-\frac{1}{2^{n+1}}$. This implies that $-\frac{2}{n}-\frac{1}{2^n} \leq -\frac{2}{n+1}-\frac{1}{2^{n+1}}$, so that $a_n = 2-\frac{2}{n}-\frac{1}{2^n} \leq 2-\frac{2}{n+1}-\frac{1}{2^{n+1}} = a_{n+1}$.
The sequence is bounded above (by 2):
Since $0 > -\frac{2}{n}-\frac{1}{2^n}$ for all $n>0$, we see that $2 > 2-\frac{2}{n}-\frac{1}{2^n}$ for all n.

36. This sequence diverges. To prove this, we'll show that it is eventually bigger than any (large) number:
Fix a large $M>0$. Then, if $N > M+1$, $a_N = N - \frac{1}{N} > (M+1)-1 = M$. That is, given any preassigned number M, the sequence is larger than M for all n after $M+1$. Thus, the sequence "runs off to infinity," i.e., it diverges.

38. This sequence converges to 0. To prove this, recall that $\lim_{n\to\infty}(k)^n = 0$ whenever $|k| < 1$. Now, our sequence $a_n = \frac{2^n-1}{3^n} = (\frac{2}{3})^n - (\frac{1}{3})^n$, and each of the terms goes to 0, so their difference goes to 0.

44. The instructions for this exercise ask us to use the result in exercise 41, an analog to Theorem 1. So, here goes.
The sequence is non-increasing:
$2^{2n+1} \le 2^{2n+2} + 1$ for all $n > 0$, so $-2^{2n+1} \ge -2^{2n+2} - 1$. Thus, $2 - 2^{2n+1} \ge 1 - 2^{2n+2}$, whence $2(1-4^n) \ge (1-4^{n+1})$, and $a_n = \frac{1-4^n}{2^n} \ge \frac{1-4^{n+1}}{2^{n+1}} = a_{n+1}$.
The sequence is not bounded below. To prove this, I'll show that the sequence eventually is less than any number:
Fix a large number $M < 0$. Since we know that $2^n > n$ for any $n > 0$, if $N > -M+1$, $a_N = \frac{1-4^N}{2^N} = (\frac{1}{2})^N - 2^N < 1 - 2^{-M+1} < 1 - (-M+1) = M$. Thus, the sequence is eventually less than any preassigned number, so it is not bounded below.

§8.2: More on Sequences

8. This sequence diverges: $\lim_{n\to\infty} \frac{1-n^3}{70-4n^2} = \lim_{n\to\infty} \frac{1/n^2-n}{70/n^2-4} = \infty$.

14. This sequence converges to 0. Note that $|-\frac{1}{2}| < 1$. Thus, $\lim_{n\to\infty}(-\frac{1}{2})^n = 0$, by a formula in table 8.1. If the fact that this oscillates back and forth over zero bothers you, note that each time it does that, it gets closer and closer to zero, so the net effect is that it's shrinking in.

20. This sequence converges to 0. Since $0 \le \sin^2(n) \le 1$ for all n, $0 \le \frac{\sin^2(n)}{2^n} \le \frac{1}{2^n}$ for all n. Thus, since $\lim_{n\to\infty} \frac{1}{2^n} = 0$, the squeeze theorem allows us to conclude that $\lim_{n\to\infty} \frac{\sin^2(n)}{2^n} = 0$.

44. This sequence converges to $\frac{1}{e}$. To see this, note that $a_n = (\frac{n}{n+1})^n = (\frac{n+1}{n})^{-n} = ((\frac{n+1}{n})^n)^{-1}$. Then $\lim_{n\to\infty} a_n = (\lim_{n\to\infty}(\frac{n+1}{n})^n)^{-1} = (e)^{-1}$, from table 8.1, again.

60. This sequence converges to -2.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \frac{1}{\sqrt{n^2-1} - \sqrt{n^2+n}} \\
&= \lim_{n\to\infty} \frac{\sqrt{n^2-1} + \sqrt{n^2+n}}{(n^2-1) - (n^2-n)} \\
&= \lim_{n\to\infty} \frac{\sqrt{n^2-1} + \sqrt{n^2+n}}{-1-n} \\
&= \lim_{n\to\infty} \frac{\sqrt{1-1/n^2} + \sqrt{1+1/n}}{-1/n-1} \\
&= -2.
\end{aligned}$$