HW 7 SOLUTIONS

MATH 122

§8.3: Series

12. This is the difference of two geometric series:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{5}{2^n} - \frac{1}{3^n} &= 4 + (\frac{5}{2} - \frac{1}{3}) + (\frac{5}{4} - \frac{1}{9}) + (\frac{5}{8} - \frac{1}{27}) + \cdots \\ &= \left(\frac{5}{1-(1/2)}\right) - \left(\frac{1}{1-(1/3)}\right) \\ &= 10 - \frac{3}{2} = \frac{17}{2}.\end{aligned}$$

18. Using partial fractions, we see $\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2}$, so $2n+1 = (A+C)n^3 + (2A+B+C+D)n^2 + (A+2B)n + B$. Thus, $B = 1$, $A = 0$, $D = -1$, and $C = 0$. Now, we can try to find the kth partial sum:

$$\begin{aligned}s_k &= \sum_{n=1}^{k} \frac{2n+1}{n^2(n+1)^2} \\ &= \sum_{n=1}^{k} \frac{1}{n^2} - \frac{1}{(n+1)^2} \\ &= (1 - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{9}) + \cdots + (\frac{1}{k^2} - \frac{1}{(k+1)^2}) \\ &= 1 - \frac{1}{(k+1)^2}\end{aligned}$$

Thus, $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} 1 - \frac{1}{(k+1)^2} = 1.$

24. $\sum_{n=0}^{\infty} (\sqrt{2})^n$ diverges, since it is a geometric series with $|r| > 1$.

30. $\sum_{n=1}^{\infty} \ln(\frac{1}{n})$ diverges by the nth term test. $a_n = \ln(\frac{1}{n}) = -\ln(n)$, so $\lim_{n\to\infty} a_n = -\lim_{n\to\infty} \ln(n) = -\infty$. Clearly, $-\infty \neq 0$, so the series doesn't converge.

36. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges, again by the nth term test. In this case, $a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdots n}{1 \cdot 2 \cdots n} > n$, and $\lim_{n\to\infty} n = \infty$, so $\lim_{n\to\infty} a_n = \infty$.

76. The area of each semicircle in row n is $(\frac{1}{2})\pi(\frac{1}{2^n})^2$, so the total area in row n is $2^n(\frac{1}{2})\pi(\frac{1}{2^n})^2 = \frac{\pi}{2}(\frac{1}{2})^n$. Thus, if we add all of the rows, we get:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\pi}{2}(\frac{1}{2})^n &= \frac{\pi}{4}\sum_{n=1}^{\infty}(\frac{1}{2})^{n-1} \\ &= \frac{\pi}{4}(\frac{1}{1-(1/2)}) = \frac{\pi}{2}.\end{aligned}$$

§8.4: The integral test

12. $\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$ diverges since it fails the nth term test. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^n}{4^n+3} = \lim_{n\to\infty} \frac{\ln(5)5^n}{\ln(4)4^n} = \frac{\ln(5)}{\ln(4)} \lim_{n\to\infty}(\frac{5}{4})^n = \infty$.

18. $\sum_{n=1}^{\infty}(1+\frac{1}{n})^n$ diverges because it fails the nth term test. $\lim_{n\to\infty}(1+\frac{1}{n})^n = e \neq 0$.

20. $\sum_{n=1}^{\infty} \frac{1}{(\ln(3))^n}$ converges, since it is a geometric series with $|r| = \frac{1}{\ln(3)} < 1$.

28. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges, using the integral test:

$$\begin{aligned}\int_1^{\infty} \frac{x}{x^2+1}\,dx &= \frac{1}{2}\int_2^{\infty}\frac{du}{u} \\ &= \frac{1}{2}\lim_{b\to\infty}[\ln(u)]_2^b \\ &= \infty,\end{aligned}$$

where I used $u = x^2+1$.