HW 8 SOLUTIONS

MATH 122

§8.5: Comparison Tests

10. The cleanest (i.e., most mathematically rigorous) way to do this is to use the LCT with $\sum \frac{1}{n}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{\ln(n)^2}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{n}{\ln(n)^2} \\
&= \lim_{n\to\infty} \frac{1}{\frac{2\ln(n)}{n}} \\
&= \lim_{n\to\infty} \frac{n}{2\ln(n)} \\
&= \lim_{n\to\infty} \frac{1}{\frac{2}{n}} \\
&= \lim_{n\to\infty} \frac{n}{2} = \infty
\end{aligned}$$

Thus, since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{\ln(n)^2}$ diverges, by the LCT, part 3.

12. Here, we'll use the LCT with $\sum \frac{1}{n^2}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{\ln(n)^3}{n^3}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{\ln(n)^3}{n} \\
&= \lim_{n\to\infty} \frac{\frac{3}{n}\ln(n)^2}{1} \\
&= \lim_{n\to\infty} \frac{3\ln(n)^2}{n} \\
&= \lim_{n\to\infty} \frac{\frac{6}{n}\ln(n)}{1} \\
&= \lim_{n\to\infty} \frac{6\ln(n)}{n} \\
&= \lim_{n\to\infty} \frac{6}{n} = 0
\end{aligned}$$

Thus, since $\sum \frac{1}{n^2}$ converges, $\sum \frac{\ln(n)^3}{n^3}$ converges, by the LCT, part 2.

18. This one is easy, provided you did # 10 correctly. We'll use the LCT with # 10 (if you don't like the answer below, you can use the LCT with $\sum \frac{1}{n}$ instead):

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{1}{1+\ln(n)^2}}{\frac{1}{\ln(n)^2}} &= \lim_{n\to\infty} \frac{\ln(n)^2}{1+\ln(n)^2} \\ &= \lim_{n\to\infty} \frac{\frac{2}{n}\ln(n)}{\frac{2}{n}\ln(n)} = 1\end{aligned}$$

Thus, since $\sum \frac{1}{\ln(n)^2}$ diverges, $\sum \frac{1}{1+\ln(n)^2}$ diverges by the LCT, part 1.

24. $\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n}$ has $a_n = \frac{3^{n-1}+1}{3^n} = \frac{1}{3} + \frac{1}{3^n}$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3} + \frac{1}{3^n} = \frac{1}{3} \neq 0$. Thus, this series fails the nth term test and, therefore, diverges.

28. We'll use the LCT to compare this series with $\sum \frac{1}{n^2}$ (since the higest power in the numerator is n^3, while the higest power in the denominator is n^5):

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{5n^3-3n}{n^2(n-2)(n^2+5)}}{\frac{1}{n^2}} &= \lim_{n\to\infty} \frac{5n^3-3n}{(n-2)(n^2+5)} \\ &= \lim_{n\to\infty} \frac{5n^3-3n}{n^3-2n^2+5n-10} \\ &= \lim_{n\to\infty} \frac{5-\frac{3}{n^2}}{1-\frac{2}{n}+\frac{5}{n^2}-\frac{10}{n^3}} = 5\end{aligned}$$

The (possibly) mysterious step in the above set of equations is the fact that I divided the numerator and denominator by n^3. Consequently, if the highest powers of n are equal in the numerator and denominator, the limit as $n \to \infty$ is always just the ratio of the coefficients. If you don't already know this, you should memorize it! You shouldn't ever have to use l'Hôpital's rule on a rational function. Back to the series, however, since $\sum \frac{1}{n^2}$ converges, $\sum \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$ converges, by the LCT, part 1.

36. For this series, we'll use the DCT. Certainly, $1 + 2^2 + 3^2 + \cdots + n^2 \geq n^2$, so $\frac{1}{1+2^2+\cdots+n^2} \leq \frac{1}{n^2}$. However, $\sum \frac{1}{n^2}$ converges, so, by the DCT, $\sum \frac{1}{1+2^2+3^2+\cdots+n^2}$ converges.

§8.6: The Ratio Test and the nth Root Test

12. We'll use the nth root test for this one:

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[n]{\left(\frac{\ln(n)}{n}\right)^n} \\ &= \lim_{n\to\infty} \frac{\ln(n)}{n} \\ &= \lim_{n\to\infty} \frac{1}{n} = 0\end{aligned}$$

Thus, since $\rho = 0 < 1$, the series converges, by the nth root test.

18. For variety, I'll use the ratio test, although you can certainly use the nth root test; on several of these types of series, the test you choose is just a matter of taste (or whim):

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{(n+1)^3}{e^{n+1}}}{\frac{n^3}{e^n}} \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \cdot e^{-1} \\
&= e^{-1} \cdot \left(\lim_{n\to\infty} \frac{n+1}{n}\right)^3 \\
&= e^{-1} \cdot 1^3 = e^{-1}
\end{aligned}$$

Thus, since $\rho = e^{-1} < 1$, the ratio test proves that this series converges.

26. For this one, either test is again sufficient. I'll do the nth root test:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n^3 2^n}} \\
&= \lim_{n\to\infty} \frac{3}{2} \cdot n^{-3/n} \\
&= \frac{3}{2} \left(\lim_{n\to\infty} n^{1/n}\right)^{-3} \\
&= \frac{3}{2} \cdot 1^{-3} = \frac{3}{2}
\end{aligned}$$

In this case $\rho = \frac{3}{2} > 1$, so the nth root test proves that the series diverges.

34. This one threw a lot of people off. If you try the ratio test, it's inconclusive. So, what's a person to do? Well, you have to notice a couple of things:
(1) Since $a_1 = \frac{1}{2}$, a_n is always strictly positive (never zero).
(2) For $n > e^{10}$, $n + \ln(n) > n + 10$, at which point $\frac{n+\ln(n)}{n+10} > 1$, so $a_{n+1} \geq a_n$
Thus, after $n > e^{10}$, $a_n > 0$ and $a_{n+1} \geq a_n$. So, in particular, $a_n \nrightarrow 0$. Therefore, the series fails the good ol' nth term test.

38. The ratio test works wonders on this one:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{(3(n+1))!}{(n+1)!(n+2)!(n+3)!}}{\frac{(3n)!}{n!(n+1)!(n+2)!}} &= \lim_{n\to\infty} \frac{(3n+3)!n!(n+1)!(n+2)!}{(3n)!(n+1)!(n+2)!(n+3)!} \\
&= \lim_{n\to\infty} \frac{(3n+1)(3n+2)(3n+3)}{(n+1)(n+2)(n+3)} \\
&= \lim_{n\to\infty} \frac{27n^3 + 54n^2 + 33n + 6}{n^3 + 6n^2 + 11n + 6} = 27
\end{aligned}$$

The last equality is due to the fact that the highest powers of n are equal in the numerator and the denominator, and 27 is the ratio of their coefficients.

This, by the way, always holds (see the solution to # 28, above). Consequently, $\rho = 27 > 1$, so the ratio test tells us that this series diverges.

44. There are several ways to do this one, as I showed in section. I think the best is to just get your hands dirty with things written as they are, using the ratio test:

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{[1\cdot 3\cdots(2n-1)\cdot(2n+1)]}{[2\cdot 4\cdots(2n)\cdot(2n+2)](3^{n+1}+1)}}{\frac{[1\cdot 3\cdots(2n-1)]}{[2\cdot 4\cdots(2n)](3^n+1)}} &= \lim_{n\to\infty} \frac{(2n+1)(3^n+1)}{(2n+2)(3^{n+1}+1)} \\ &= 1\cdot \lim_{n\to\infty} \frac{1+3^{-n}}{3+3^{-n}} = \frac{1}{3}\end{aligned}$$

So, $\rho = \frac{1}{3} < 1$, and the series converges.