HW 9 SOLUTIONS

MATH 122

§8.7: Absolute and Conditional Convergence

38. This series converges absolutely by the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\frac{((n+1)!)^2 3^{n+1}}{(2n+3)!}}{\frac{(n!)^2 3^n}{(2n+1)!}} \\
&= \lim_{n\to\infty} \frac{n!(n+1)n!(n+1)\cdot 3}{n!n!(2n+2)(2n+3)} \\
&= \lim_{n\to\infty} \frac{3(n+1)^2}{4n^2+10n+6} = \frac{3}{4}
\end{aligned}$$

Thus, $\rho = \frac{3}{4} < 1$, so the series $\sum |a_n|$ converges. Hence, the series converges absolutely.

42. This series converges conditionally. To see that $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges, note that it satisfies all of the conditions of the Alternating Series Test. We show below that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ doesn't converge:

$$\begin{aligned}
n \geq 1 \quad &\Rightarrow \quad 3n > 1 \Rightarrow 4n > 1+n \\
&\Rightarrow \quad 2\sqrt{n} > \sqrt{n+1} \\
&\Rightarrow \quad 3\sqrt{n} > \sqrt{n+1} + \sqrt{n} \\
&\Rightarrow \quad \frac{1}{3\sqrt{n}} < \frac{1}{\sqrt{n}+\sqrt{n+1}}
\end{aligned}$$

Thus, since $\frac{1}{3}\sum \frac{1}{\sqrt{n}}$ is a divergent p-series with $p < 1$, we know that $\sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges by the DCT.

You may ask, "Where in the world did this come from?" The thinking behind this is that $\sqrt{n}+\sqrt{n+1}$ is a lot like $2\sqrt{n}$, but a tiny bit bigger, so you can guess that it's not bigger than $3\sqrt{n}$ and see if it works. We know that $\sum \frac{1}{\sqrt{n}}$ diverges, so it seems reasonable that this one will diverge, too.