The Lorenz attractor exists

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Abstract

We prove that the Lorenz equations support a strange attractor, as conjectured by Edward Lorenz in 1963. We also prove that the attractor is robust, i.e., it persists under small perturbations of the coefficients in the underlying differential equations. The proof is based on a combination of normal form theory and rigorous numerical computations.

1 Introduction

The following non-linear system of differential equations,

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 &= \varrho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2, \end{aligned} \tag{1}$$

was introduced 1963 by Edward Lorenz, see [4]. As a crude model of atmospheric dynamics, these equations led Lorenz to the discovery of sensitive dependence of initial conditions - an essential factor of unpredictability in many systems. Numerical simulations for an open neighbourhood of the classical parameter values $\sigma = 10, \beta = 8/3$ and $\varrho = 28$ suggest that almost all points in phase space tend to a strange attractor $\mathcal{A}$ - *the Lorenz attractor.* Based on numerical data, a geometric model describing the dynamics of the flow was introduced by Guckenheimer and Williams, see [2], [7]. We prove that this model does indeed give an accurate description of the dynamics of (1).

By the use of a Poincaré section, the flow of (1) can be reduced to a *first return map* R acting on the section $\Sigma \subset \{x_3 = \varrho - 1\}$, as schematically illustrated in Figure 1.

Note that R is not defined on the line $\Gamma = \Sigma \cap W^s(0)$: these points tend to the origin, and never return to Σ. Due to the fixed point at the origin, the return times are not bounded. This constitutes a serious obstruction to any numerical approach. This is overcome by introducing a local change of coordinates, and we prove the following properties of the return map R:

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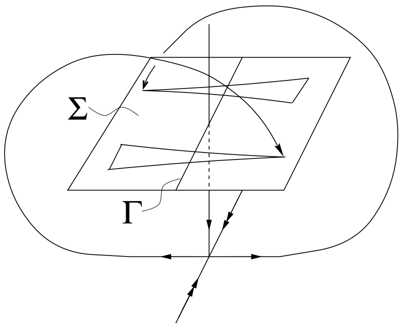

Figure 1: The return map acting on Σ. [L'application de retour agissant sur Σ.]

- There exists a compact set $N \subset \Sigma$ such that $N \setminus \Gamma$ is *forward invariant* under R, i.e., $R(N \setminus \Gamma) \subset \mathrm{int}(N)$. This ensures that the flow has an attracting set $\mathcal{A}$ with a large basin of attraction. We can then form a cross-section of the attracting set: $\mathcal{A} \cap \Sigma = \cap_{n=0}^{\infty} R^n(N) = \Lambda$.
- On N, there exists a cone field $\mathfrak{C}$ which is mapped strictly into itself by DR, i.e., for all $x \in N$, $DR(x) \cdot \mathfrak{C}(x) \subset \mathfrak{C}(R(x))$. The cones of $\mathfrak{C}$ are centered along two curves which approximate Λ, and each cone has an opening of $10°$.
- The tangent vectors in $\mathfrak{C}$ are eventually expanded under the action of DR: there exists $C > 0$ and $\lambda > 1$ such that for all $v \in \mathfrak{C}(x)$, $x \in N$, we have $|DR^n(x)v| \geq C\lambda^n|v|$, $n \geq 0$. In fact, the expansion is strong enough to ensure that R is topologically transitive on Λ, see below.

These three properties of R verify the conjecture made by Lorenz 35 years ago. With some further efforts, see [6], we can also say something about the statistical properties of the flow.

Theorem: *For the classical parameter values, the Lorenz equations support a robust strange attractor $\mathcal{A}$. Furthermore, the flow admits a unique SRB measure μ_X with* $\mathrm{supp}(\mu_X) = \mathcal{A}$.

The proof can be broken down into two main sections: one global part, which involves rigorous computations, and one local part, which is based on normal form theory. The novelty of the method of proof lies in that, rather than producing a traditional mathematical proof, we construct an algorithm which, if successfully executed, proves the existence of the strange attractor. This algorithm is put into effect via a C-program. The source codes and the list of initial data that are used in the proof are available at `http://www.math.uu.se/~warwick/thesis.html`.

2 Local theory and normal forms

In this section we will construct a change of variables $\xi = \zeta + \phi(\zeta)$ which, in a small cube centered at the origin, transforms the Lorenz equations $\dot{\xi} = A\xi + F(\xi)$ (here in Jordan normal form) into a carefully selected normal form, which is virtually linear (although it is crucial that it need not be completely linear). Inside the cube, we can then estimate the evolution of trajectories analytically, and thereby we avoid the problem of having to use computers in regions where the flow times are unbounded. Although there exist C^k-linearization theorems, see e.g. [1], we choose to use a method introduced by H. Poincaré. This method is based on an analytic change of coordinates, and even though it introduces a small divisor problem, we feel that the desired estimates are more straightforward to obtain using this method.

We will combine multi-index notation $a_n\zeta^n = a_{n_1,n_2,n_3}\zeta_1^{n_1}\zeta_2^{n_2}\zeta_3^{n_3}$ and vector notation $\zeta = (\zeta_1, \zeta_2, \zeta_3)$. Also, we define $\zeta^n \in \mathcal{O}^{10}(\zeta_1) \cap \mathcal{O}^{10}(\zeta_2, \zeta_3)$ if $n_1 \geq 10$ and $n_2 + n_3 \geq 10$. In this section we will work in a complex neighbourhood of the origin, and use the following max-norms:

$$|\zeta| = \max\{|\zeta_i| : i = 1,2,3\} \qquad \|f\|_r = \max\{|f(\zeta)| : |\zeta| \le r\}.$$

The main estimates are summarized in the following proposition:

Proposition 2.1 *There exists a close to identity change of variables $\xi = \zeta + \phi(\zeta)$ with*

$$\|\phi\|_r \le \frac{r^2}{2} \qquad r \le 1$$

such that the Lorenz equations, $\dot{\xi} = A\xi + F(\xi)$, are transformed into the normal form $\dot{\zeta} = A\zeta + G(\zeta)$, where $G(\zeta) \in \mathcal{O}^{10}(\zeta_1) \cap \mathcal{O}^{10}(\zeta_2, \zeta_3)$, and satisfies

$$\|G\|_r \le 7 \cdot 10^{-9} \frac{r^{20}}{1-3r} \qquad r < \frac{1}{3}.$$

Seeing that the change of variables and its inverse are analytic, we can use Cauchy-Riemann estimates to gain information on their derivatives. Analogously, we can get estimates of the derivatives of G. This allows us to estimate the exit of any trajectory or tangent vector entering the cube.

By splitting the 3-space of natural numbers into two disjoint sets: $\mathbb{N}^3 = \mathbb{U}_{10} \cup \mathbb{V}_{10}$, where

$$\mathbb{V}_{10} = \{(n_1, n_2, n_3) \in \mathbb{N}^3 : n_1 < 10 \text{ or } n_2 + n_3 < 10\},$$

we may define the following filters, which act on formal vector-valued polynomials $f(\zeta) = \sum_n \alpha_n \zeta^n$:

$$\{f(\zeta)\}_{\mathbb{U}_{10}} = \sum_{n \in \mathbb{U}_{10}} \alpha_n \zeta^n, \quad \text{and} \quad \{f(\zeta)\}_{\mathbb{V}_{10}} = \sum_{n \in \mathbb{V}_{10}} \alpha_n \zeta^n.$$

Our Ansatz is to do the calculations with formal vector-valued polynomials, and solve for ϕ by direct substitution. We arrive at the following functional equation:

$$L_A\phi(\zeta) = F(\zeta + \phi(\zeta)) - D\phi(\zeta)G(\zeta) - G(\zeta), \tag{2}$$

where $L_A\phi(\zeta) = D\phi(\zeta)A\zeta - A\phi(\zeta)$. The operator L_A is linear, and it leaves the spaces of homogeneous vector-valued polynomials of any degree invariant. On component level, we have $L_{A,i}(\zeta^n) = (n\lambda - \lambda_i)\zeta^n$. By filtering (2), we get for $i = 1,2,3$

$$L_{A,i}\phi_i(\zeta) = \{F_i(\zeta + \phi(\zeta))\}_{\mathbb{V}_{10}} \quad \text{and} \quad G_i(\zeta) = \{F_i(\zeta + \phi(\zeta))\}_{\mathbb{U}_{10}} - \sum_{j=1}^{3} \frac{\partial \phi_i}{\partial \zeta_j}(\zeta) G_j(\zeta). \tag{3}$$

The left-most equation can be formally solved by a power series

$$\phi_i(\zeta) = \sum_{|n|=2}^{\infty} a_{i,n}\zeta^n \qquad (i = 1,2,3)$$

provided none of the appearing *divisors* $n\lambda - \lambda_i$ $(i = 1,2,3)$ vanish. The fact that any ζ^n appearing in ϕ must satisfy $n \in \mathbb{V}_{10}$ is crucial for avoiding high-order resonances. It allows us to prove the following computer-aided lemma, which gives the existence of a formal series for ϕ.

Lemma 2.2 *For any $n \in \mathbb{V}_{10}$ with $|n| \ge 2$, the divisors $n\lambda - \lambda_i$ $(i = 1,2,3)$ are bounded away from zero. Furthermore, for $|n| \ge 58$, there exists a sharp lower bound on the modulus of these divisors:*

$$|n\lambda - \lambda_i| \ge \big|9\lambda_1 + \big(|n| - 9\big)\lambda_3 - \lambda_i\big| \qquad (i = 1,2,3).$$

The proof requires estimates of 19,386 low-order divisors. These are computed by a small C-program, `smalldiv.c`. The sharp lower bound on the higher-order divisors is an analytic result.

Knowing that the formal power series for ϕ defined by (3) is well defined, we want to show that it actually converges. By using standard techniques of majorants, see e.g [5], the problem is reduced to proving the convergence of a single variable power series $\psi(r) = \sum_{k=2}^{\infty} c_k r^k$ satisfying $\|\phi\|_r \leq \psi(r)$. The coefficients of ψ are given by the following recursive scheme

$$c_k r^k = \frac{5}{\Omega(k)} \left[\left(r + \sum_{i=2}^{k-1} c_i r^i \right)^2 \right]_k \qquad (k = 2, 3, ...), \tag{4}$$

where $[\sum \alpha_n \zeta^n]_k = \alpha_k \zeta^k$, and $\Omega(k) = \min\{|\lambda n - \lambda_i| \colon |n| = k, n \in \mathbb{V}_{10}, i = 1, 2, 3\}$. Due to the recursiveness, the first coefficients have a large effect on the radius of convergence. In order to enlarge this radius, we postpone the use of (4). Instead, we estimate the 186,576 first coefficients $a_{i,n}$ of ϕ (computed by another small C-program, `coeff.c`), and set $c_k = \sum_{|n|=k} \max_{i=1,2,3} |a_{i,n}|$ for $k = 2, ..., 70$ before using (4). This gives a considerably better estimate on the bound of $\|\phi\|_r$. Using similar techniques on the right-most part of (3), we get also get a bound on the normal form G. Here the situation is somewhat simpler as there are no divisors present.

3 Rigorous numerics

First, we present our candidate for the trapping region N. This set consists of two disjoint components, N^- and N^+, each made up of 350 adjacent rectangles belonging to the return plane $x_3 = 27$ $(= \varrho - 1)$. We will call these small rectangles $N_i^{\pm}$, and write

$$N = N^- \cup N^+ = \big(\bigcup_{i=1}^{350} N_i^-\big) \cup \big(\bigcup_{i=1}^{350} N_i^+\big).$$

The two components of N have the same symmetry as the Lorenz equations, i.e., $N_i^+ = S(N_i^-)$, where $S(x_1, x_2, x_3) = (-x_1, -x_2, x_3)$. Thanks to this symmetry, we only have to perform the computations on one component of N. When it is not relevant which component we are considering, we omit the $\pm$ labeling of the small rectangles.

Dealing with one N_i at a time, we compute a pseudo-path that strictly contains the flow of N_i. The pseudo-paths are obtained by introducing several intermediate return planes $\Pi^{(m)}$, which are either x_1x_2-planes or x_2x_3-planes, according to whether $|\dot{x}_1| \leq |\dot{x}_3|$ or vise-versa, respectively. The initial rectangle N_i is flowed to the first plane $\Pi^{(1)}$ by using an Euler method with rigorous error estimates. In the plane $\Pi^{(1)}$, we take the rectangular hull of the largest image of N_i, giving us a new starting rectangle $\mathcal{R}^{(1)}$. This rectangle is then flowed to $\Pi^{(2)}$ and so on. If a rectangle $\mathcal{R}^{(m)}$ has grown too large it is partitioned into smaller rectangles, which are then treated separately. This whole procedure is repeated until we return to Σ from above, as illustrated in Figure 1. Due to the contracting forces, the pseudo-return of N_i will consist of many overlapping rectangles $Q_{i,j}$ $j = 1, \ldots, k(i)$, whose union strictly contains $R(N_i)$.

The use of rectangles significantly simplifies the computations: when flowing between two intermediate planes $\Pi^{(m)}$, $\Pi^{(m+1)}$, it is generically the corners of $\mathcal{R}^{(m)}$ that yield the largest rectangular hull $\mathcal{R}^{(m+1)} \subset \Pi^{(m+1)}$. This fact allows us to reduce the error analysis to small pieces of $\mathcal{R}^{(m)}$, which greatly reduces the local errors. With only finite precision, however, this property becomes "pseudo-generic", and has to be confirmed at every stage. The exceptional cases are treated slightly different.

Turning to the question concerning the cone field, we define the field by equipping each N_i with an initial cone. Each cone is represented by the two angles α_i^-, α_i^+ its boundary vectors

$u^{(0)}, v^{(0)}$ make with the x_1-axis. As mentioned before, we always take $\theta^{(0)} = \alpha_i^+ - \alpha_i^- = \pi/18$. We then use similar techniques as just described: when a rectangle has been flowed from $\Pi^{(m)}$ to $\Pi^{(m+1)}$, we are provided with a box containing the path of the rectangle. The algorithm also gives us upper and lower bounds on the flow time involved. By solving the nine equations governing the partial derivatives of the flow, we obtain rigorous bounds on the evolution of the tangent vectors flowing through the box. By translating the flowed vectors onto the intermediate plane $\Pi^{(m+1)}$, and by selecting (incorporating the errors) the pair of vectors $u^{(m+1)}, v^{(m+1)}$ making the largest angle $\theta^{(m+1)}$, we ensure that the resulting cone contains all images of tangent vectors from the initial cone. At the return, each rectangle $Q_{i,j}$ is thus equipped with a cone represented, as above, by two angles $\beta_{i,j}^-, \beta_{i,j}^+$, $j = 1, \dots, k(i)$.

When computing the minimal expansion in each cone, we start with the widest pair of vectors, $u^{(m)}, v^{(m)}$ at each intermediate plane $\Pi^{(m)}$, as described above. If $\theta^{(m+1)} \leq \theta^{(m)}$, the minimal expansion $\varepsilon^{(m)}$ is attained on the boundary of the cone, i.e., $\varepsilon^{(m)}$ is the smallest growth factor of the images of $u^{(m)}, v^{(m)}$. If $\theta^{(m+1)} > \theta^{(m)}$, however, we must adjust this estimate by a factor which is quadratically close to unity in $\theta^{(m+1)}$. At the return each rectangle $Q_{i,j}$ is equipped with an expansion estimate $\mathcal{E}_{i,j} = \Pi_{m=0}^{n(i,j)} \varepsilon_{i,j}^{(m)}$, and $\mathcal{E}_i = \min_j \mathcal{E}_{i,j}$ gives an estimate for all vectors of the cone associated with N_i.

One major advantage of our numerical method is that we totally eliminate the problem of having to control the global effects of rounding errors due to the computer's internal floating point representation. This is achieved by using a high-dimensional analogue of *interval arithmetic*. Each object Ξ (e.g. a rectangle or a tangent vector) subjected to computation is equipped with a maximal absolute error Δ_Ξ, and can thus be represented as a box $\Xi \pm \Delta_\Xi = [\Xi_1 - \Delta_{\Xi_1}, \Xi_1 + \Delta_{\Xi_1}] \times \cdots \times [\Xi_n - \Delta_{\Xi_n}, \Xi_n + \Delta_{\Xi_n}]$. When following an object from one intermediate plane to another, we compute upper bounds on the images of $\Xi_i + \Delta_{\Xi_i}$, and lower bounds on the images of $\Xi_i - \Delta_{\Xi_i}$, $i = 1, \dots, n$. This results in a new box $\tilde{\Xi} \pm \Delta_{\tilde{\Xi}}$, which strictly contains the exact image of $\Xi \pm \Delta_\Xi$. To ensure that we have strict inclusion, we use quite rough estimates on the upper and lower bounds. This gives us a margin which is much larger than any error caused by rounding possibly could be. Hence, the rounding errors are taken into account in the computed box $\tilde{\Xi} \pm \Delta_{\tilde{\Xi}}$, and we may continue to the following intermediate plane by restarting the whole process.

As long as we do not flow close to a fixed point, the local return maps are well defined diffeomorphisms, and the computer can handle all calculations. Some rectangles, however, will approach the origin (which is a fixed point), and then the computations must be interrupted as discussed in the previous section.

4 Conclusions

The main C-program verifies that the union $\cup_{j=1}^{k(i)} Q_{i,j}$ is strictly contained in N for each i, and thus we conclude that $R(N \setminus \Gamma) \subset \text{int}(N)$, as desired. It also checks that if $Q_{i,j} \cap N_k \neq \emptyset$, then $\alpha_k^- < \beta_{i,j}^- < \beta_{i,j}^+ < \alpha_k^+$. This proves the existence of a forward invariant cone field: for all $x \in N$, $DR(x) \cdot \mathfrak{C}(x) \subset \mathfrak{C}(R(x))$.

We now turn to the question of expansion. Much to our surprise, we found regions in N which were contracted in all directions under R. In view of the geometric model, which is uniformly expanding in one direction, this was not anticipated. We prove, however, that all tangent vectors within the cone field are *eventually* expanded under DR. More precisely, given any orbit $x_0, x_1, \dots$, where $x_j = R^j(x_0)$, we can divide it into non-overlapping pieces $[x_0, \dots, x_{k_0}], [x_{k_0+1}, \dots, x_{k_1}], \dots$ where all but the first piece accumulate an expansion factor greater than two. This value is crucial for the conclusion of transitivity, see [7]. We also show that $k_{i+1} - k_i \leq 29$, which gives a very crude lower estimate of the positive Lyapunov exponent: $\lambda > \sqrt[29]{2} > 1.024$. To prove these claims, we use some of the output from the computer runs to produce both forward and backward pseudo-orbits of R with associated expansion estimates.

Since the flow of (1) is uniformly volume-contracting and transversal to N, a finite iterate of the return map R is area-contracting on N. This property together with the existence of a forward invariant unstable cone field implies that R admits an invariant stable foliation with $C^{1+\alpha}$ leaves, see §3 in [3]. The singular map f induced by taking quotients along the stable leaves acts on an interval $I = [-a, a]$, and satisfies the following properties:

- The restriction of f to $[-a, 0)$ and $(0, a]$ is of class $C^{1+\alpha}$ with $f'(x) \geq K > 0$ for all $x \neq 0$.
- There exists $C > 0$, $\lambda > 1$ such that $(f^n)'(x) \geq C\lambda^n$, for all $n \geq 0$.
- For any interval $J \subset I$ there exists $n \geq 0$ such that $f^n(J) = I$.

It follows that f admits a unique finite SRB measure μ_f with $\mathrm{supp}(\mu_f) = I$. From this measure it is possible to construct an SRB measure μ_R with $\mathrm{supp}(\mu_R) = \Lambda$ for the return map R, and also an SRB measure μ_X for the flow, see [6].

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