Math 105, Fall 2004 – Solutions to Prelim 2

1.

(a) Henri has to take 3 bottles from 5+7=12, and the order in which they are taken doesn't matter. Therefore, there are

$$\binom{12}{3} = \frac{12!}{9!\,3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9!}{9!\,3!} = \frac{12 \cdot 11 \cdot 10}{3!} = \frac{1320}{6} = 220$$

possible choices.

(b) Henri has to choose 1 bottle from the 5 bottles of red and 2 bottles from the 7 bottles of white. By the multiplication principle, there are

$$\binom{5}{1} \cdot \binom{7}{2} = 5 \cdot 21 = 105$$

ways of doing this.

2.

Note that $S = \{0, 1, 2, \ldots, 98, 99\}$ is the sample space for this problem.

(a) $E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$.
$F = \{9, 18, 27, 36, 45, 54, 63, 72, 81, 90\}$.
Assuming equally likely outcomes, we have:

$$P(E) = \frac{n(E)}{n(S)} = \frac{20}{100} = \frac{1}{5}$$

and

$$P(F) = \frac{n(F)}{n(S)} = \frac{10}{100} = \frac{1}{10}.$$

(b) Assuming equally likely outcomes,

$$P(E \cap F) = \frac{n(E \cap F)}{n(s)} = \frac{n(\{9, 18\})}{100} = \frac{2}{100} = \frac{1}{50}.$$

So by definition of conditional probability, we have:

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/50}{1/10} = \frac{1}{5}.$$

Hence $P(E) = \frac{1}{5} = P(E|F)$, and so E and F are independent events.

(c) $G = \{5, 14, 23, 32, 41, 50\}$. So assuming equally likely outcomes, we have

$$P(G) = \frac{n(G)}{n(S)} = \frac{6}{100} = \frac{3}{50}$$

and

$$P(E \cap G) = \frac{n(E \cap G)}{n(S)} = \frac{n(\{5, 14\})}{100} = \frac{2}{100} = \frac{1}{50}.$$

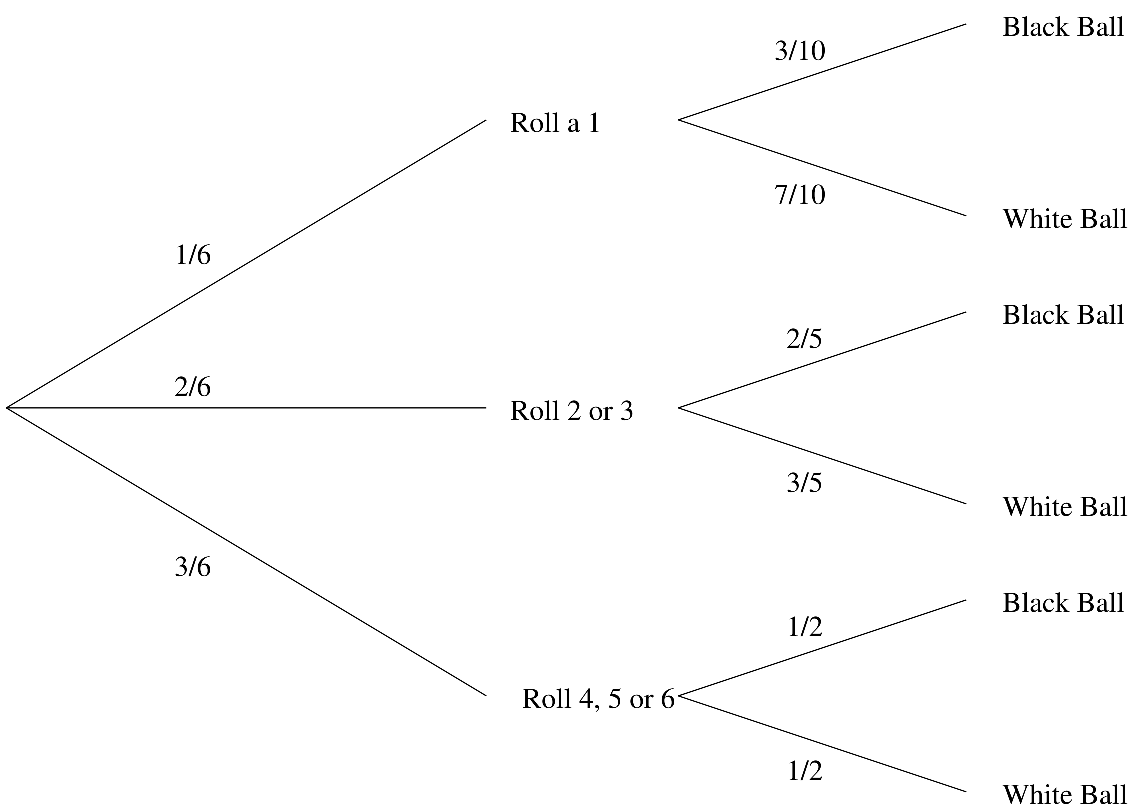
Hence

$$P(G|E) = \frac{P(E \cap G)}{P(E)} = \frac{1/50}{1/5} = \frac{1}{10}.$$

So $P(G|E) = \frac{1}{10} \neq \frac{3}{50} = P(G)$ and therefore E and G are *not* independent.

3.

(a)



(b) From the tree digram, we see that the probability of getting a black ball is

$$\frac{1}{6} \cdot \frac{3}{10} + \frac{1}{3} \cdot \frac{2}{5} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{60} + \frac{8}{60} + \frac{15}{60} = \frac{26}{60} = \frac{13}{30}.$$

(c) Let B be the event "getting a black ball" and T be the event "rolling a 2 or a 3" (note this is the same as "picked ball from second jar"). Then by definition of conditional probability,

$$P(T|B) = \frac{P(T \cap B)}{P(B)} = \frac{P(B|T)P(T)}{P(B)} = \frac{(1/3)(2/5)}{13/30} = \frac{4}{13}\,.$$

Let E be the event “rolled a two”. Then similarly, we have

$$P(E|B) = \frac{P(E \cap B)}{P(B)} = \frac{P(B|E)P(E)}{P(B)} = \frac{(1/6)(2/5)}{13/30} = \frac{2}{13}\,.$$

4.

(a) Converting percentages into probabilities, we have

$$P(A) = 0.51 \text{ and } P(F|A) = 0.58.$$

(b) Using (the special case of) Bayes' Theorem and noting that $L = A'$, we have

$$\begin{aligned} P(A|F) &= \frac{P(F|A) \cdot P(A)}{P(F|A) \cdot P(A) + P(F|A') \cdot P(A')} \\ &= \frac{P(F|A) \cdot P(A)}{P(F|A) \cdot P(A) + P(F|L) \cdot P(L)} \\ &= \frac{(0.58)(0.51)}{(0.58)(0.51) + (0.45)(0.49)} \\ &= \frac{0.2958}{0.2958 + 0.2205} \approx 0.573\,. \end{aligned}$$

(c)

$$\begin{aligned} P(F) &= P(F|A) \cdot P(A) + P(F|A') \cdot P(A') \\ &= P(F|A) \cdot P(A) + P(F|L) \cdot P(L) \\ &= 0.2958 + 0.2205 = 0.5163 \end{aligned}$$

So $0.5163 \times 100\% \approx 51.6\%$ of voters in this election were women.

5.

(a) The 9 books can be arranged in

$$P(9, 9) = 9! = 362,880 \text{ ways.}$$

(b) The math books can be arranged in 3! ways, the physics books can be arranged in 4! ways, and the chemistry books can be arranged in 2! ways. There are 3! ways to choose the order of the 3 groups of books. Therefore, using the multiplication principle, the number of possible arrangements is

$$3!4!2!3! = 6 \cdot 24 \cdot 2 \cdot 6 = 1728\,.$$

(c) Note that the type of book (math, physics, chemistry) does not matter for this part of the question. To give one book away to each of his friends, Henri must take 4 books from 9, counting order. There are

$$P(9,4) = \frac{9!}{(9-4)!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5!} = 9 \cdot 8 \cdot 7 \cdot 6 = 3024$$

ways of doing this.

(d) Observe that the number of ways of doing this is

$$\begin{aligned} \text{answer to part (c)} \quad &- \quad \text{number of ways of giving exactly one non-math book to each friend} \\ &= \quad P(9,4) - P(6,4) = 3024 - 360 = 2664. \end{aligned}$$

6.

Let E be the event of getting a "Jack in the basement". There are:

- $\binom{4}{1}$ ways of choosing a Jack;
- $P(12,2)$ ways of choosing 2 values (other than Jack) for the pair and the single card;
- $\binom{4}{2}$ ways of choosing a pair of cards of a given value;
- $\binom{4}{1}$ ways of choosing a single card of a given value; and
- $\binom{52}{4}$ ways of choosing 4 cards from a deck of 52.

Note that in choosing 2 values other than Jack for the pair and single card, order is important because a King and two Queens is different from a Queen and two Kings, for example. So using the multiplication principle, we have

$$n(E) = \binom{4}{1} \cdot P(12,2) \cdot \binom{4}{2} \cdot \binom{4}{1} = 4 \cdot (12 \cdot 11) \cdot 6 \cdot 4 = 12,672\,.$$

Assuming equally likely outcomes, we have

$$P(E) = \frac{n(E)}{n(S)} = \frac{12,672}{270,725} \approx 0.0468\,.$$

7.

We assume equally likely outcomes throughout this problem. Note that $n(S)$ is the number of ways Billy can choose 6 marbles from 15, which is $\binom{15}{6} = 5005$.

(a) Let E denote the event in question. There are $\binom{5}{3}$ ways of choosing 3 red marbles, $\binom{5}{2}$ ways of choosing 2 blue marbles, and $\binom{5}{1}$ ways of choosing 1 green marble. So by the multiplication principle,

$$n(E) = \binom{5}{3} \cdot \binom{5}{2} \cdot \binom{5}{1} = 10 \cdot 10 \cdot 5 = 500\,.$$

Hence

$$P(E) = \frac{n(E)}{n(S)} = \frac{500}{5005} = \frac{100}{1001} \approx 0.0999\,.$$

(b) Let F denote the event in question. There are $\binom{5}{4}$ ways of choosing 4 blue marbles, and $\binom{10}{2}$ of choosing 2 marbles of a different color. So by the multiplication principle,

$$n(F) = \binom{5}{4} \cdot \binom{10}{2} = 5 \cdot 45 \;= 225\,.$$

Hence

$$P(E) = \frac{n(E)}{n(S)} = \frac{225}{5005} = \frac{45}{1001} \approx 0.0450\,.$$

(c) Let G denote the event in question. Then

$$\begin{aligned}
P(G) &= P(\text{billy grabs exactly 4 blue marbles}) + P(\text{billy grabs exactly 5 blue marbles}) \\
&\quad\ (\text{these events are mutually exclusive}) \\
&= \frac{\binom{5}{4} \cdot \binom{10}{2}}{\binom{15}{6}} + \frac{\binom{5}{5} \cdot \binom{10}{1}}{\binom{15}{6}} \\
&= \frac{5 \cdot 45 + 1 \cdot 10}{5005} = \frac{235}{5005} = \frac{47}{1001} \approx 0.0470\,.
\end{aligned}$$

(d) Let H denote the event in question. Then

$$\begin{aligned}
P(H) &= P(F) + P(\text{grabbing 4 green marbles}) + P(\text{grabbing exactly 4 red marbles}) \\
&\quad\ (\text{these events are mutually exclusive}) \\
&= 3 \cdot P(F) \ (\text{by symmetry}) \\
&= 3 \cdot \frac{45}{1001} = \frac{135}{1001} \approx 0.1349\,.
\end{aligned}$$