Prelim 1 Solutions

Problem 1 (Total 10 points)
Give the least square line that has the best fit to the following data points:

$$(0,2),\ (1,-2),\ (2,-2) \text{ and } (3,2).$$

Hint: Recall that the slope and y-intercept of the best fit line are given by the formulas

$$nb + \left(\sum_{i=1}^{n} x_i\right) m = \sum_{i=1}^{n} y_i \quad \text{and} \quad \left(\sum_{i=1}^{n} x_i\right) b + \left(\sum_{i=1}^{n} x_i^2\right) m = \sum_{i=1}^{n} x_i y_i.$$

<u>**Solution:**</u>
We begin by computing the sums required for the equations given in the hint:

$$\sum_{i=1}^{n} x_i = 6 \quad \sum_{i=1}^{n} y_i = 0 \quad \sum_{i=1}^{n} x_i^2 = 14 \quad \sum_{i=1}^{n} x_i y_i = 0$$

Plugging these values into the equations gives

$$(4)\,b + (6)\,m = 0 \quad \text{and} \quad (6)\,b + (14)\,m = 0$$

Solving these equations by any method yields the unique solutions $m = 0$ and $b = 0$. Therefore the best fit line is $y = 0$, a horizontal line through the origin.

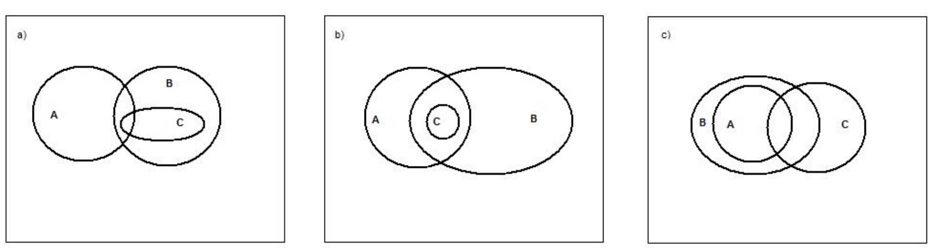
Problem 2 (Total 16 points)
Draw Venn diagrams that satisfy the following requierments.
Let $A \subseteq U$, $B \subseteq U$, and $C \subseteq U$ and

(a) **(5 points)** $A \cap B \neq \emptyset$, $C \subseteq B$ and $C \cap A \neq \emptyset$.

(b) **(5 points)** $A \cap B \neq \emptyset$ and $C \subset A \cap B$.

(c) **(6 points)** $A \subseteq B$, $C \cap B \neq \emptyset$, and $C \cap A' \neq \emptyset$.

<u>**Solution:**</u> (Other satisfactory solutions exist...)



Problem 3 (Total 16 points)

(1) Give the equations for the lines that go through the following pairs of points. Write all equations in point slope form.

  (a) **(3 points)** $(-1, 3)$ and $(2, 4)$.

  (b) **(3 points)** $(-6, -2)$ and $(9, 3)$.

  (c) **(3 points)** $(1, 2)$ and $(-4, 2)$.

(2) **(3 points)** Which of the following lines above are parallel?

(3) **(4 points)** Give and equation for the line that is perpendicular to the line passing trough the points $(-6, -2)$ and $(9, 3)$

<u>**Solution:**</u>

(1a) $m = \frac{4-3}{2-(-1)} = \frac{1}{3}$ thus, we have $y - 3 = \frac{1}{3}(x + 1)$ or $y - 4 = \frac{1}{3}(x - 2)$

(1b) $m = \frac{3-(-2)}{9-(-6)} = \frac{1}{3}$ thus, we have $y + 2 = \frac{1}{3}(x + 6)$

(1c) $m = \frac{2-2}{1-(-4)} = 0$ thus, we have $y - 2 = 0(x - 1)$ or $y = 2$

(2) Clearly the lines in parts (a) and (b) have the same slope ($m = \frac{1}{3}$).

(3) Recall perpendicular lines have *negative reciprocal slopes* ($m_1 = -\frac{1}{m_2}$). Thus any line with slope $m = -3$ suffices.

Problem 4 (Total 25 points)

Let $\mathcal{A}$ be the following matrix

$$\mathcal{A} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

(a) **(10 points)** Find the multiplicative inverse matrix $\mathcal{A}^{-1}$ of matrix $\mathcal{A}$ using an augmented matrix.

(b) **(5 points)** Compute $\mathcal{A}^{-1}\mathcal{A}$ and $\mathcal{A}\mathcal{A}^{-1}$.

(c) **(10 points)** Solve the following system of linear equations using the inverse matrix $\mathcal{A}^{-1}$ calculated above.

$$\begin{cases} x - 2y = 1 \\ x + y - z = 1 \\ z - x = 1 \end{cases}$$

Solution:
First, we find the inverse by writing the augmented matrix

$$\mathcal{A} = \left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Using the the Gauss-Jordan method, we manipulate the $\mathcal{A}$ part of the matrix until we have transformed it to the identity, which yields

$$\mathcal{A} = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right]$$

The right hand side "keeps track" of all our manipulations (in a sense), such that under this method whenever one multiplies the original matrix $\mathcal{A}$ by the $3x3$ matrix in the right hand side above, we should get whatever is in the left hand side... That is, for the augmented matrix above, the right hand side should be $\mathcal{A}^{-1}$. Thus,

$$\mathcal{A}^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

For part (b), we just multiply the matrices $\mathcal{A}$ and $\mathcal{A}^{-1}$ to show they are inverses. For example, matrix multiplication gives...

$$\mathcal{A}{\cdot}\mathcal{A}^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+0+0 & 2-2+0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}_{3\mathrm{x}3}$$

and similarly, we show that

$$\mathcal{A}^{-1} \cdot \mathcal{A} = \mathrm{I}_{3\mathrm{x}3}$$

For part (c), we solve the equations by first noticint that we can consider the equation in matrix form (Check this with matrix multiplication and compare to the equations!):

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We can solve for our unknowns by right-multiplying both sides of this equality by $\mathcal{A}^{-1}$, and simplifying the right hand side using matrix multiplication (see text and notes if needed!). Thus,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$$

Problem 5 (Total 8 points)
Find all solutions for the following system of linear equations.

$$\begin{cases} 4x - 2y = 4 \\ 2x = 3 + y \end{cases}$$

<u>**Solution:**</u> Rewriting the equations yields

$$\begin{aligned} 4x - 2y &= 4 \\ 2x - y &= 3 \end{aligned}$$

The Echelon, Gauss-Jordan, or any other similar method will suffice from this point on. For example, we can attempt to solve these equations by writing them in matrix form:

$$\begin{bmatrix} 4 & -2 & 4 \\ 2 & -1 & 3 \end{bmatrix} \longrightarrow 2R_2 - R_1 \text{ in } R_2 \longrightarrow \begin{bmatrix} 4 & -2 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

Putting these numbers back into equation form, we see that since $0 \neq 2$ the equations are inconsistent and there are *no solutions*.
Alternatively, solving for y in the second equation gives $y = 2x - 3$ which, when substituted into the other equation, yields "$3 = 4$". This is clearly false, and thus we conclude that there are no solutions.

Problem 6 (Total 15 points)
Find all solutions to the following system of linear equations using the Gauss-Jordan method.

$$\begin{cases} x + 2w + 1 = 0 \\ x + 3 = -y \\ y = 2w - 2 \end{cases}$$

<u>**Solution:**</u>
Here we needed to use Gauss-Jordan, so we begin writing the equations in matrix form. Various versions exist depending on the ordering of the variables (here we use w then x then y)... but all all yield what are essentially the same result.

$$\begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & -3 \\ -2 & 0 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 1 & 1 & -3 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Putting this back into equation form, we can move all the y terms to the right hand side and solve for w and x in terms of y, which yields $w = 1 + \frac{y}{2}$, $x = -3 - y$, and $y = y$. [Note these are easy to check by considering the last two of the given equations.] So there are *infinitely many solutions* which are given by

$$w = 1 + \tfrac{y}{2} \qquad x = -3 - y \qquad \text{and} \qquad y = y.$$

Problem 7 (Total 10 points)
An on campus survey was conducted among 150 students, 31 faculty and 19 staff to determine what condiments Cornellians were putting on their veggie burgers. The following results were observed.

```
67 people liked BBQ sauce                   157 people liked Ketchup
100 people liked Ketchup and Mustard only   3 people liked Mustard only
7 people liked Ketchup and BBQ sauce only   30 people liked Ketchup only
123 people like mustard
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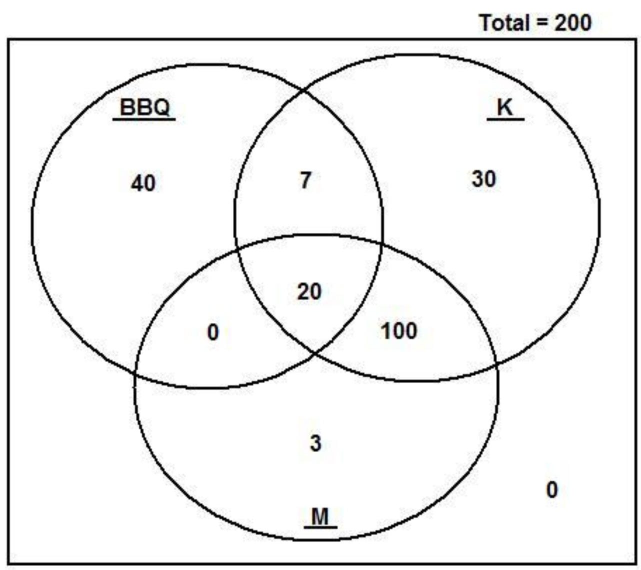
(a) **(4 points)** Draw a Venn Diagram illustrating the results of this survey that indicates the number of people that like and dislike all combinations of condiments.

(b) **(2 points)** How many participants don't like any condiments?

(c) **(2 points)** How many participants liked two or more condiments on their burgers?

(d) **(2 points)** How many liked BBQ sauce and mustard only?

<u>**Solution:**</u>



(b) 0

(c) $20 + 0 + 7 + 100 = 127$

(d) 0