Math 105 Prelim #3 – November 29, 2005

1. (*25 points*) For each of the following matrices, indicate whether or not it can be the transition matrix of a Markov chain and, if so, whether the corresponding Markov chain is regular. Explain your answer.

$$A = \begin{pmatrix} 0 & 1 \\ 1/3 & 2/3 \end{pmatrix}; \quad B = \begin{pmatrix} 1/3 & 2/3 \\ 0 & 1 \end{pmatrix}; \quad C = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/4 & 2/3 & 1/12 \\ 1/5 & 3/4 & 1/10 \end{pmatrix}.$$

2. (*25 points*) At the Johnson family Thanksgiving, there are 15 place settings. Each setting has a 3/5 chance of having an orange napkin and a 2/5 chance of having a tan napkin.

a) What is the probability that there are exactly 10 orange napkins? (You do not need to simplify your expression.)

b) Use the normal distribution to approximate the probability of having at least 7 orange napkins.

3. (*25 points*)

a) Suppose the weights of *ExtraordiNuts*^{TM} brand peanuts follow a normal distribution with mean $\mu = 2.5$ grams and standard deviation $\sigma = .25$ grams. While splitting a bag of *ExtraordiNuts*^{TM}, Bob proposes that he and Alice play the following game: They select a peanut at random from the bag and weigh it. If it weighs more than 3 grams, Bob gives Alice 2 dollars. Otherwise, Alice gives Bob 1 dollar. What are Alice's expected winnings from the game?

b) Suppose Alice, having tired of peanuts, picks ten cards randomly with replacement from a standard 52-card deck. What is the probability that at most 3 of the chosen cards are hearts? (You do not have to simplify your expression.)

4. (*25 points*) Each year Aunt Mildred brings either pumpkin cookies or cranberry bread to the Johnson family Thanksgiving. She can never remember what she brought more than one year ago. If she brings pumpkin cookies one year, she has a 4/5 probability of bringing cranberry bread the next year. If she brings cranberry bread one year, she has 2/3 probability of bringing pumpkin cookies the next year. This determines a regular Markov chain with two states: namely, the first state occurs when Aunt Mildred brings pumpkin cookies; the second when she brings cranberry bread.

a) Draw a transition diagram and give the transition matrix for this Markov chain.

b) Suppose that Aunt Mildred brings pumpkin cookies this year. What is the probability that she will bring pumpkin cookies two years from now?

c) Find the equilibrium vector for this Markov chain without using your calculator.

Math 105: Solutions to Prelim 3

December 2, 2005

1) Both A and B can be matrices for Markov chains, since each has entries between 0 and 1, and each row in each matrix sums to 1. C could not be the matrix for a Markov chain, since the sum of its third row is $1/5 + 3/4 + 1/10 = 21/20 \neq 1$. The Markov chain associated with A is regular, since

$$A^2 = \begin{pmatrix} 1/3 & 2/3 \\ 2/9 & 7/9 \end{pmatrix}$$

has all non-zero entries. However, since

$$B^2 = \begin{pmatrix} 1/9 & 8/9 \\ 0 & 1 \end{pmatrix}$$

has a zero in the same place as B, we know that B^k has a zero in that place for all $k > 0$. Therefore, the Markov chain associated with B is not regular.

2a) Let 'success' be the event that a place setting receives an orange napkin. Then we can view the 15 place settings as 15 trials of a binomial experiment, where $p = 3/5$. Thus, by the binomial probability formula, the probability of exactly 10 successes (i.e., 10 orange napkins) is:

$$\binom{15}{10}(3/5)^{10}(2/5)^5$$

2b) This binomial distribution has mean $\mu = np = (15)(3/5) = 9$ and standard deviation $\sigma = \sqrt{np(1-p)} = \sqrt{(15)(3/5)(2/5)} \simeq 1.897$. Let P denote probability in the normal distribution with the above mean and standard deviation. Then the probability of having at least 7 orange napkins is approximated by $P(x > 6.5)$. If P_s denotes probability in the standard normal distribution, then $P(x > 6.5) = P_s(z > \frac{6.5-9}{1.897}) = P_s(z > -1.32) = 1 - P_s(z < -1.32)$. Consulting the table, we see that $P_s(z < -1.32) = .0934$. So the probability of having at least 7 orange napkins is approximately $1 - .0934 = .9066$.

3a) Let P be probability in the normal distribution with mean $\mu = 2.5$ grams and standard deviation $\sigma = .25$, and let P_s be probability in the standard normal distribution. Then the probability that a randomly selected peanut from a bag of *ExtraordiNuts*^{TM} is $P(x > 3) = P_s(z > \frac{3-2.5}{.25}) = P_s(z > 2) = 1 - P_s(z < 2)$. Consulting the table, we have $P(x > 3) = 1 - .9772 = .0228$. Let X be the random variable that assigns, to each outcome of the experiment, the amount of money (in dollars) Alice wins. Then X assumes the value 2 if the chosen peanut weighs more than 3 grams (which happens with probability .0228), and it assumes the value -1 otherwise. Therefore, Alice's expected winning from playing the game once is:

$$E(X) = (2)(.0228) + (-1)(.9772) = -.9316$$

meaning Alice loses, on average, 93 cents a game.

3b) Since each card is replaced after being picked, the experiment described is binomial. The probability of selecting a heart is $p = 1/4$. The experiment is repeated $n = 10$ times. Therefore, the probability that Alice draws at most 3 hearts is:

$$\binom{10}{0}(1/4)^0(3/4)^{10}+\binom{10}{1}(1/4)^1(3/4)^9+\binom{10}{2}(1/4)^2(3/4)^8+\binom{10}{3}(1/4)^3(3/4)^7$$

4a) The transition matrix for what Aunt Mildred brings to Thanksgiving dinner is as follows, where the first state is pumpkin cookies and the second is cranberry bread:

$$P = \begin{pmatrix} 1/5 & 4/5 \\ 2/3 & 1/3 \end{pmatrix}$$

b) Aunt Mildred bringing pumpkin cookies one year can be represented by the probability vector $v = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Thus the probability vector after two iterations of the Markov chain is $vP^2 = (vP)P = \begin{pmatrix} 1/5 & 4/5 \end{pmatrix} P =$

$\begin{pmatrix} 43/75 & 32/75 \end{pmatrix}$. Since the first state represents Pumpkin cookies, the probability that Aunt Mildred brings Pumpkin cookies two years from now is just the first entry of the probability vector obtained above, namely 43/75.

c) Suppose $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ is an equilibrium vector for the Markov chain. Then $vP = v$, namely $\begin{pmatrix} v_1 & v_2 \end{pmatrix}\begin{pmatrix} 1/5 & 4/5 \\ 2/3 & 1/3 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$. This gives us two equations: $(1/5)v_1 + (2/3)v_2 = v_1$, and $(4/5)v_1 + (1/3)v_2 = v_2$. Since v must be a probability vector, it must also satisfy $v_1 + v_2 = 1$. This last equation gives us $v_2 = 1 - v_1$. Plugging in this into our first equation gives: $(1/5)v_1 + (2/3)(1 - v_1) = v_1 \Rightarrow v_1 = 5/11$. Since $v_1 + v_2 = 1$, $v_2 = 6/11$. Therefore, $v = \begin{pmatrix} 5/11 & 6/11 \end{pmatrix}$.