MATH 105: Assignment 11 Solutions

Each problem worth 4 points

10.1.12

Yes, $\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is a transition matrix since

- It is a square matrix
- Entries are between 0 and 1.
- Sum of entries in each row is 1.

10.1.18

The transition matrix associated to this diagram is

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} .6 & .2 & .2 \\ .9 & .02 & .08 \\ .4 & 0 & .6 \end{bmatrix} \end{array}.$$

Note that it is indeed a transition matrix because:

- It is a square matrix
- Entries are between 0 and 1.
- Sum of entries in each row is 1.

10.1.22

$$D = \begin{bmatrix} .3 & .2 & .5 \\ 0 & 0 & 1 \\ .6 & .1 & .3 \end{bmatrix}, \quad D^2 = \begin{bmatrix} .39 & .11 & .455 \\ .6 & .1 & .3 \\ .36 & .15 & .49 \end{bmatrix}, \quad D^3 = \begin{bmatrix} .41 & .12 & .45 \\ .36 & .15 & .49 \\ .40 & .12 & .48 \end{bmatrix}.$$

The probability that state 1 changes to state 2 after three repetitions is .12.

10.1.30

First note that the probability of a G_1 becoming a G_2 must be .2 so that the row sum is 1. Now we can write out the transition matrix.

$$\begin{array}{c|c} & \begin{matrix} G_0 & G_1 & G_2 \end{matrix} \\ \begin{matrix} G_0 \\ G_1 \\ G_2 \end{matrix} & \begin{bmatrix} .85 & .1 & .05 \\ 0 & .8 & .2 \\ 0 & 0 & 1 \end{bmatrix} \end{array}.$$

10.1.34

a Let A denote agricultural, U denote urban, and I denote idle. Then P is given by

$$\begin{array}{c|c} & \begin{matrix} A & U & I \end{matrix} \\ \begin{matrix} A \\ U \\ I \end{matrix} & \begin{bmatrix} .80 & .15 & .05 \\ 0 & .9 & .1 \\ .1 & .2 & .7 \end{bmatrix} \end{array}.$$

b The initial distribution for the land is given by

$$X_0 = \begin{bmatrix} .35 & .1 & .55 \end{bmatrix}.$$

c and **d** These are obtained through multiplication by the appropriate power of P. Namely, after ten years the land use pattern will be

$$X_0P = [.335\ .2525\ .4125],$$

and after twenty years it will be

$$X_0P^2 = [.30925\ .36\ .33075]$$

e The transition matrix for a 20-yr period, which you will have used for part **d** is:

$$\begin{bmatrix} .645 & .265 & .09 \\ .01 & .83 & .16 \\ .15 & .335 & \boxed{.515} \end{bmatrix}.$$

f This probability can be read off of P^2. It is the boxed number.

10.2.10 Let P be $\begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}$, and let $V = [v_1\ v_2]$ denote the equilibrium vector.

We must solve the matrix equation

$$[v_1\ v_2]\begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix} = [v_1\ v_2].$$

This matrix equation leads us to solve the following system of equations (after a rearrangement),

$$\begin{array}{rcl} -.2v_1 + .1v_2 & = & 0 \\ .2v_1 - .1v_2 & = & 0 \end{array}.$$

You will usually find it more convenient to get rid of the decimals, and that's what we'll do here. This system of equations can be written in matrix form as:

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right].$$

We now use the row-echelon method. We try to change the 2 in the second equation to a 0. Clearly, we should add both equations. We obtain

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

(If you have done things correctly you should always get a row of zeros at this point.) The solutions (*there are many*) to this system are

$$\begin{array}{rcl} v_2 & = & t \\ v_1 & = & \frac{v_2}{2} = \frac{t}{2} \end{array}, \qquad \text{where } t \text{ can be any number}$$

or in vector form

$$[\frac{t}{2}\ t], \qquad \text{with } t \text{ any number .} \tag{1}$$

Finally, to get *a probability vector* we use the fact that the entries of the solution must add up to 1. That is, we must find *the* t such that $t + \frac{t}{2} = 1$. Therefore, $t = \frac{1}{3}$.

Hence, pluggin in $t = \frac{1}{3}$ in (1) we see that the equilibrium vector for P is

$$[\frac{1}{3}\ \frac{2}{3}].$$

10.2.12

The method is exactly the same as in the previous exercise. Denoting the matrix by P and the equilibrium vector by v. We try to find v so that

(a) $vP = v$

(b) the sum of the entries of v equals 1.

After rearranging (**a**) turns into the following system of equations,

$$\begin{array}{rcl} -5v_1 + 1v_2 + 2v_3 & = & 0 \\ 2v_1 - 6v_2 + 2v_3 & = & 0 \\ 3v_1 + 5v_2 - 4v_3 & = & 0 \end{array} . \tag{2}$$

Now, in order to use the row-echelon method, we rewrite the system in matrix form. To save space, we will not keep the column of zeros since this one stays a column of zeros when we "row-echelon" our system.

$$\begin{bmatrix} -5 & 1 & 2 \\ 2 & -6 & 2 \\ 3 & 5 & -4 \end{bmatrix}.$$

Now we apply the row-echelon method

$$\begin{bmatrix} -5 & 1 & 2 \\ 2 & -6 & 2 \\ 3 & 5 & -4 \end{bmatrix} \to \begin{bmatrix} -5 & 1 & 2 \\ 0 & -28 & 14 \\ 0 & 28 & -14 \end{bmatrix} \to \begin{bmatrix} -5 & 1 & 2 \\ 0 & -28 & 14 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

(If you have done things correctly you should always get a row of zeros at this point.) In order to obtain *all* the solutions to the system you proceed as follows. Set $v_3 = t$, then go to the second equation and find what v_2 is in terms of v_3 (which is the same as t). In our case, we see that $v_2 = \frac{v_3}{2}$, so $v_2 = \frac{t}{2}$. Finally, we figure out what v_1 is using the first equation. We see that $v_1 = \frac{1}{5}v_2 + \frac{2}{5}v_3$, hence, in terms of t we have $v_1 = \frac{1}{5}\frac{t}{2} + \frac{2}{5}t = \frac{t}{2}$.

Therefore, the solution to (2) is, in vector form:

$$[\frac{t}{2}\ \frac{t}{2}\ t], \qquad \text{where } t \text{ is any number.} \tag{3}$$

Finally, we use **(b)** in order to obtain only one vector. Hence, we want to find *the* t such that $\frac{t}{2} + \frac{t}{2} + t = 1$. This happens for $t = \frac{1}{2}$. Therefore, the equilibrium vector is obtained after plugging in $t = \frac{1}{2}$ into (3). That is, the equilibrium vector is:

$$[\frac{\frac{1}{2}}{2}\ \frac{\frac{1}{2}}{2}\ \frac{1}{2}] = [\frac{1}{4}\ \frac{1}{4}\ \frac{1}{2}]$$

10.2.30 The transition matrix is P $=$ $\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} .3 & .3 & .4 \\ .15 & .3 & .55 \\ .3 & .6 & .1 \end{bmatrix} \end{array}.$

What we are asked to do is to find the equilibrium vector, v, for the matrix P. Hence, we need to find v satisfying the following.

(a) $vP = v$

(b) the sum of the entries of v equals 1.

The method that we will follow will be different from the one in the previous exercise. Namely, we will combine conditions **(a)** and **(b)** from the beginning. If one does that then one gets a system with 4 equations. After rearranging (and getting rid of the decimals) one gets

$$\begin{array}{rcll} v_1 + v_2 + v_3 & = & 1 & \text{extra equation because we use } \textbf{(a)} \text{ now} \\ -70v_1 + 15v_2 + 30v_3 & = & 0 & \\ 3v_1 - 7v_2 + 6v_3 & = & 0 & \\ 40v_1 + 55v_2 - 90v_3 & = & 0 & \end{array} \tag{4}$$

This system can be solved using the row-echelon method. The solution to (4), which is the equilibrium vector is

$$[\frac{60}{251}\ \frac{102}{251}\ \frac{89}{251}].$$

Therefore, the fraction of mice in A in the long-range is $\frac{60}{251}$, in B it is $\frac{102}{251}$ and in C it is $\frac{89}{251}$.

10.2.38

In order to answer this question we have to find the equilibrium vector.

This is done exactly as in **10.2.10**. In this case the equations that you have to solve are

$$\begin{array}{rcl} -88v_1 + 54v_2 & = & 0 \\ 88v_1 - 54v_2 & = & 0, \end{array}$$

whose solution is $[\frac{54}{88}t\ t]$ for t any number.

To obtain a probability vector you choose t such that $\frac{54}{88}t + t = 1$. This happens only for $t = \frac{88}{142}$. Hence, the equilibrium vector is

$$[\frac{54}{88}\frac{88}{142}\ \frac{88}{142}] = [\boxed{\frac{27}{71}}\ \frac{44}{71}].$$

Since $\frac{27}{71} \approx .38$, we see that about 38% of letters in English text are expected to be vowels.