MATH 105 Prelim 2 and Solutions- Fall 2007

Problem 1 - (15 total points)

Consider an experiment where you roll two fair dice and add the two values together.

(a) **(5 points)** Write down the sample space of the experiment above.

Solution: All possible sums of 2 dice give $\mathbb{S} = \{2, 3, ..., 12\}$.

(b) **(5 points)** Show that the the sample space is not equally likely by computing the probabilities of the outcomes 12, 11 and 10.

Solution: Looking at the outcome of the possible rolls of two dice, we see that there is 1 way to get a 12, 2 ways to get an 11 (5,6 or 6,5), and 3 ways to get a 10 (4,6; 5,5 or 6,4). Since there are 6x6 equally likely outcomes (pairs of numbers) that are possible, we have that

$$\mathbb{P}(10) = \tfrac{3}{36} = \tfrac{1}{13} \qquad \mathbb{P}(11) = \tfrac{2}{36} = \tfrac{1}{18} \qquad \mathbb{P}(12) = \tfrac{1}{36}$$

(c) **(5 points)** What is the probability of getting a sum which is less than or equal to 9?

Solution: $\mathbb{P}(\leq 9) = 1 - \mathbb{P}(\geq 10) = 1 - \mathbb{P}(10) - \mathbb{P}(11) - \mathbb{P}(12) = 1 - \frac{3}{36} - \frac{2}{36} - \frac{1}{36} = \frac{5}{6}$

Problem 2 - (17 total points)

A class has 10 boys and 12 girls. People in the class have either long or short hair. 25% of the girls have short hair and 20% of the boys have long hair. Calculate the following probabilities.

Solution: Note that the following problems are made easier by first computing the probability (or count) of the 4 "intersections": "short haired boy", "short haired girl", "long haired boy", and "long haired girl" (which are 8, 3, 2, and 9 respectively).

(a) **(5 points)** The probability that a person from the class has long hair?

Solution: $\mathbb{P}(\text{long}) = \frac{(.75)(12)+(.2)(10)}{12+10} = \frac{11}{22} = \frac{1}{2}$

(b) **(5 points)** The probability that a person from the class is a girl?

Solution: $\mathbb{P}(\text{girl}) = \frac{12}{22} = \frac{6}{11}$

(c) **(6 points)** The probability that a person from the class is a girl OR has long hair?

Solution 1: $\mathbb{P}(\text{girl} \cup \text{'long hair'}) = \mathbb{P}(\text{girl}) + \mathbb{P}(\text{long haired boy}) = \frac{12}{22} + \frac{2}{22} = \frac{7}{11}$

Solution 2: -or- $\mathbb{P}(\text{girl} \cup \text{'long hair'}) = 1 - \mathbb{P}(\text{short haired boy}) = 1 - \frac{8}{22} = \frac{7}{11}$

Solution 3: -or- $\mathbb{P}(\text{girl} \cup \text{'long hair'}) = \mathbb{P}(\text{girl}) + \mathbb{P}(\text{'long hair'}) - \mathbb{P}(\text{girl} \cap \text{'long hair'})$

$$= \tfrac{12}{22} + \tfrac{11}{22} - \mathbb{P}(\text{long hair}|\text{girl})\mathbb{P}(\text{girl}) = \tfrac{12}{22} + \tfrac{11}{22} - (.75)\tfrac{12}{22} = \tfrac{7}{11}$$

(d) **(6 points)** The probability that a person from the class is a boy given that they have short hair?

Solution: $\mathbb{P}(\text{boy} \mid \text{short hair}) = \frac{\mathbb{P}(\text{short haired boy})}{\mathbb{P}(\text{short hair})} = \frac{8}{11}$

Problem 3 - (18 total points)

The population of birds in Ithaca is comprised of only three subspecies: Prairie, Hudson Bay and Coastal. 10% of the population are Prairie birds and 80% of the population are Hudson Bay birds. Birds in this population can either be orange or gray. 99% of Prairie birds are orange, 10% of Hudson Bay birds are orange and 10% of Coastal birds are orange. Calculte the following probabilities.

(a) **(6 points)** If a bird is randomly observed, what is the probability that it is a Coastal bird?

Solution: $\mathbb{P}(\text{Coastal}) = 1 - .80 - .10 = .10$

(b) **(6 points)** If an orange bird is observed, what is the probability that it is a Prairie bird?

Solution: Using Bayes Theorem,

$$\mathbb{P}(\text{Prairie}|\text{orange}) = \frac{(.99)(.10)}{(.99)(.10) + (.10)(.80) + (.10)(.10)} = \frac{(.99)}{(.99) + (.80) + (.10)} = \frac{99}{189} = \frac{9(11)}{9(21)} = \frac{11}{21}$$

(c) **(6 points)** Use your answer from part (b) to find the probability that an orange bird is either a Hudson Bay bird OR a Coastal bird.

Solution: Since "Hudson Bay or Coastal" is the complement of "Prairie" it follows that

$$\mathbb{P}(\text{Coastal OR Hudson Bay}) = 1 - \mathbb{P}(\text{Prairie}) = 1 - \frac{11}{21} = \frac{10}{21} \approx .5$$

Problem 4 - (10 points)

An experiment has possible outcomes A, B and C. Suppose that $P(A) = 1/3$, $P(B) = 3/4$ and $P(C) = 1/4$. Calculate the following probabilities.

(a) **(5 points)** Assuming that A and C are independent calculate $P(A \cup C)$.

Solution: Independence gives $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(C)$, thus

$$\mathbb{P}(A \cup C) = \mathbb{P}(A) + \mathbb{P}(C) - \mathbb{P}(A \cap C) = \mathbb{P}(A) + \mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(C) = \frac{1}{3} + \frac{1}{4} - \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) = \frac{7}{12} - \frac{1}{12} = \frac{1}{2}$$

(b) **(5 points)** Assuming that $A \subseteq B$, what is $P(A \cap B)$?

Solution: If $A \subseteq B$ then $A \cap B = A$, therefore $\mathbb{P}(A \cap B) = \mathbb{P}(A) = 1/3$

Problem 5 - (15 points)

A museum has 12 different paintings that it would like to hang in a row on the wall. 6 are portraits, 4 are landscapes and 2 are still lifes.

(a) **(5 points)** In how many ways can the paintings be placed on the wall?

Solution: The 12 paintings are all different, so there are 12! ways to hang them on the wall. 12 ways to pick the first one, times 11 ways to pick the second, ...

(b) **(5 points)** In how many ways can this be done if a still life is placed first?

Solution: First select a still life, then pick the 11 others. Multiplying gives $\binom{2}{1}(11!) = 2(11!)$ ways.

(c) **(5 points)** In how many ways can this be done if the separate groups of portraits, landscapes and still lives are each to be kept together?

Solution: Here we count up the number of arrangements by first counting up ways of arranging which groups go first, second and third. Next we multiply that by the number of ways we can arrange paintings within each group. This gives $\binom{3}{1}\binom{2}{1}\binom{1}{1}$ or 3! ways to arrange groups, and then 6!4!2! ways to re-arrange within each group. Thus there are (3!)(6!4!2!) ways to arrange them in groups.

(d) **(5 points)** A patron wants to buy a group of 6 paintings; 2 portraits, 3 landscapes and 1 still life. In how many ways can he purchase such a group of paintings?

Solution: $\binom{6}{2}\binom{4}{3}\binom{2}{1} = \frac{6!}{2!4!}\frac{4!}{3!1!}\frac{2!}{1!1!} = \frac{6 \cdot 5}{2}\frac{4}{1}\frac{2}{1} = 120.$

Problem 6 - (15 points)

A hand of 5 cards is dealt from a standard deck of 52 cards. Find the probability of the following hands.

(a) **(5 points)** Four cards of the same value and one card of a different value.

Solution: Order doesn't matter here, so to count we can arrange all possible hands so matching cards come first. We are only interested in those with 4 of a kind. We see that for each of the 13 card values (A, 2, 3, ..., 10, J, Q, K) there are 12 hands with 4 cards of that kind and a 5th of another kind (e.g. 3333A, 33332, 33334, 33334, .., 3333Q, 3333K). Therefore we can count all such hands as $\binom{13}{1}\binom{4}{4} \cdot \binom{12}{1}\binom{4}{1}$ or simply (13)(12)(4). (Note $\binom{12}{1}\binom{4}{1} = \binom{48}{1}$, so we could just as easily count $\binom{48}{1}$ ways to pick the 5th card from the remaining 48.)

Therefore the probability of getting such a hand is

$$\mathbb{P}(4 \text{ of a kind}) = \frac{(13)(12)(4)}{\binom{52}{5}}$$

(b) **(5 points)** Three cards of the same value and two cards of two different values.

Solution: Similarly,

$$\mathbb{P}(\text{a pair and 3 of a kind}) = \frac{13\binom{4}{2}12\binom{4}{3}}{\binom{52}{5}}$$

(c) **(5 points)** Two pairs, where each pair of cards has the same value, and one card is of different value.

Solution: First pick the 2 types of pairs, then pick 2 of each. Lastly, pick 1 different card.

$$\mathbb{P}(\text{2 different pair, no 3 of a kind}) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot \binom{11}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot \binom{44}{1}}{\binom{52}{5}}$$