February 19, 2008

PRELIM 1 v3 Solution v4 Math 171

Directions:

- No notes with formulas, etc. are permitted on this exam.
- Calculators are allowed and we also provide you with tables.
- All work must be shown in the answer booklet. Be sure it is complete, neat, and in order.
- Clearly indicate your final answers.
- Proper mathematical justification must be provided in order to earn full credit; correct answers with no work shown may receive no credit. Note that using the calculator does not take the place of showing your work.
- You may use standard notation; any new notation or abbreviations you use must be clearly defined.
- You have 90 minutes to complete this exam.
- Please indicate the time of your MWF class on the front of your exam book.

Problem 1 *Cards* **[16 points]** Three cards are picked at random from a deck of cards without replacement. Find the probability of each event described below:

a: You get three aces.
b: You get all red cards.
c: The second card is your only red card.
d: You have at least one red card.

(An ordinary deck has 52 *cards, with an equal number of red and black cards. There are four aces in such a deck.)*

Soln:

a: $\left(\frac{4}{52}\right)\left(\frac{3}{51}\right)\left(\frac{2}{50}\right) = .00018.$
b: $\left(\frac{26}{52}\right)\left(\frac{25}{51}\right)\left(\frac{24}{50}\right) = .1176.$
c: $\left(\frac{26}{52}\right)\left(\frac{26}{51}\right)\left(\frac{25}{50}\right) = .1275.$
d: "At least one red card" is the complement of "all black cards". The probablilty of all black cards is the same as the probability of all red cards, so using b), our answer here is $1 - .1176 = .8824$.

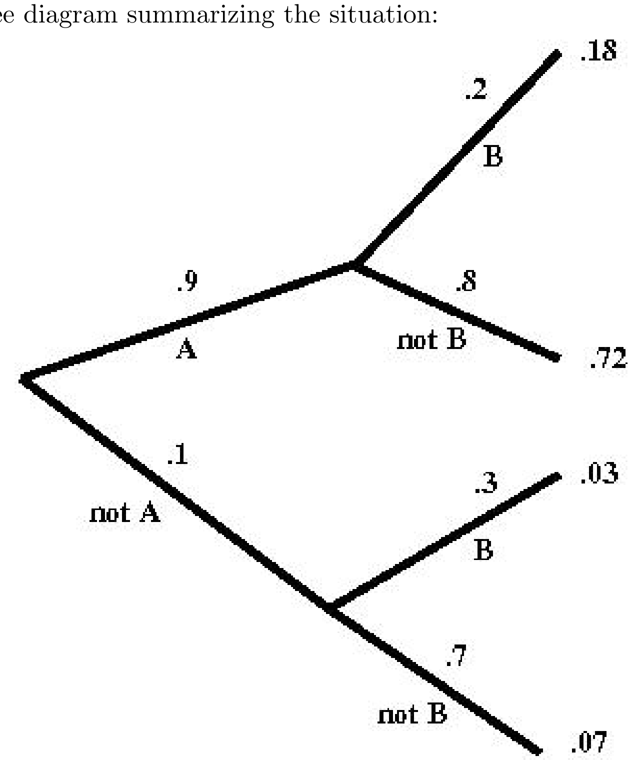
Problem 2 *Game Consoles* **[18 points]** Mr. Smith wants to buy two popular video game consoles for his two grandchildren on the morning of Black Friday. There are two GameStop stores, stores A and B, in town but each has a very limited inventory. As a result, each customer is allowed to buy only one console at each store. Mr. Smith decides to go to store A first and then rush to store B. It turns out that the probability he gets a console at store A is 90%. If he gets it at store A and spends time at the checkout, the probability he gets another one at store B is only 20%. If he does not get the console at store A, the probability he gets one at store B is 30%.

a: Are the two events, (1)getting a console at store A, and (2)getting a console at store B, independent?
b: Find the probability that Mr. Smith gets two consoles.
c: Find the probability that he gets no console.
d: Let X be a random variable describing the number of consoles Mr. Smith gets. Using your results earlier in this problem, give a probability model for X. (i.e. a table describing the probability distribution of the random variable X.)
e: Find the expected value and standard deviation of X.

Soln: Let A be the event of getting a console at store A and B the event of getting a console at store B. We are given:

$$\begin{aligned} P(A) &= .9 \\ P(B|A) &= .2 \\ P(B|\text{not } A) &= .3 \end{aligned}$$

Below is a tree diagram summarizing the situation:



a: The best solution here is to compare $P(B)$ with $P(B|A) = .2$. From the tree diagram, $P(B) = .18 + .03 = .21$ which is different from .2, and so these are not independent. (It is also basically correct to observe that this follows from $P(B|A) \neq P(B|\text{not } A)$ though this is not what our definition asked us to consider.)
b: As in the tree diagram, $P(A \cap B) = P(B|A)\ P(A) = .18$.

c: As in the tree diagram, $P(\text{not } A \cap \text{not } B) = P(\text{not } B|\text{not } A)\ P(\text{not } A) = .07$.

d: We just computed $P(X = 2)$ and $P(X = 0)$in parts b) and c). The entry for $X = 1$ is obtained either by the complement rule or form the tree diagram.

X	**probability**
0	.07
1	.75
2	.18

e:

$$\begin{aligned} E(X) &= .07(0) + .75(1) + .18(2) = 1.11 \\ Var(X) &= .07(1.11)^2 + .75(.11)^2 + .18(.89)^2 \\ &= .086 + .009 + .143 = .238 \end{aligned}$$

So $\sigma_X = \sqrt{.238} = .48$.

Problem 3 *Typing* [**17 points**] Suppose that the distribution of net typing rate in words per minute (wpm) for experienced typists can be approximated by a normal curve with mean 60 wpm and standard deviation 15 wpm. *(The paper "Effect of Age and Skill in Typing" (Journal of Experimental Psychology [1984]: 345-371) described how net rate is obtained from gross rate by using a correction for errors.)*

Please calculate your answers on this problem to at least the accuracy attainable with Table Z.

a: What is the probability that a randomly selected typist's net rate is at most 60 wpm? less than 60 wpm?

b: What is the probability that a randomly selected typist's net rate is between 45 and 90 wpm?

c: If your new secretary came from this population and he had a typing rate exceeding 105 wpm, would you be surprised? Why or why not? (Note: The largest net rate in a sample described in the paper is 104 wpm.)

d: Suppose that two typists are independently selected. What is the probability that both their typing rates exceed 75 wpm?

e: Suppose that special training is to be made available to the slowest 20 % of the typists. What typing speeds would qualify individuals for this training ?

Soln: Let X be the randomly selected typist's net rate.

a.

$$\begin{aligned} P(X \leq 60) &= P(\frac{X - 60}{15} \leq \frac{60 - 60}{15}) = P(Z \leq 0) = \Phi(0) = 0.5 \\ P(X < 60) &= 0.5 \textit{ by continuity} \end{aligned}$$

b.

$$P(45 \leq X \leq 90) \quad = \quad P(\frac{45-60}{15} \leq \frac{X-60}{15} \leq \frac{90-60}{15}) = P(-1 \leq Z \leq 2) = \Phi(2) - \Phi(-1) = 0.8186$$

c.

$$P(X \geq 105) = 1 - P(X \leq 105) = 1 - P(\frac{X-60}{15} \leq \frac{105-60}{15}) = 1 - P(Z \leq 3) = 1 - \Phi(3) = 0.0013$$

Yes, I would be surprised.
d. Let X_1 and X_2 be the net rates of the two typists, respectively. By independence,

$$\begin{aligned} P(X_1 \geq 75 \text{ and } X_2 \geq 75) &= P(X_1 \geq 75) \times P(X_2 \geq 75) \\ &= (1 - P(X_1 \leq 75)) \times (1 - P(X_2 \leq 75)) \\ &= (1 - P(Z \leq 1)) \times (1 - P(Z \leq 1)) \\ &= .0252 \end{aligned}$$

e. Let x_0 be the qualifying threshold. Since $20\% = \Phi(-0.8416)$, x_0 satisfies $P(X \leq x_0) = \Phi(-0.8416)$. On the other hand,

$$P(X \leq x_0) \quad = \quad P(\frac{X-60}{15} \leq \frac{x_0-60}{15}) = P(Z \leq \frac{x_0-60}{15}) = \Phi(\frac{x_0-60}{15})$$

Therefore,

$$\frac{x_0-60}{15} = -0.8416$$

Hence, $x_0 = 47.38$

Problem 4 *Free Throws* [**16 points**] Suppose the probability of making a free throw in basketball is 0.9 and that all attempts are independent. *(Show your work here as elsewhere on the exam. Answers to this problem may not be based on the binomialcdf function, in case your calculator has this capability.)*

- **a) :** Find the probability of making exactly five free throws in six attempts.
- **b) :** Find the probability of making at most four free throws in six attempts.
- **c) :** Calculate the expected number of successful free throws in 400 attempts. What would the standard deviation be?
- **d) :** Using normal approximation, estimate the probability of at least 350 successful free throws.

Soln:
a. Binom(6,.9) where p = .9 and q = (1-p) = .1.

$$P(X = 5) = \binom{6}{5}(.9)^5(.1)^1 = .3543$$

b.

$$\begin{aligned} P(X \le 4) &= 1 - P(X > 4) = 1 - [P(X = 5) + P(X = 6)] \\ &= 1 - \left[\binom{6}{5}(.9)^5(.1)^1 + \binom{6}{6}(.9)^6(.1)^0) \right] = .1143 \end{aligned}$$

c. $E(X) = np = 400(.9) = 360$ for 400 attempts. And $SD(X) = \sqrt{npq} = \sqrt{(400)(.9)(.1)} = 6$.
d. Want $P(X \geq 350)$. To use normal approximation verify that $np, nq \geq 10$. Clearly the condition is satisfied. So we can do Binom$(400, .9) \approx N(360, 6)$. Then $z = \frac{x-360}{6} = \frac{350-360}{6} = -\frac{5}{3} \approx -1.667$. So

$$P(X \geq 350) \approx P(z > -1.667) = 1 - P(z < -1.667) = 1 - .0479 = .9521$$

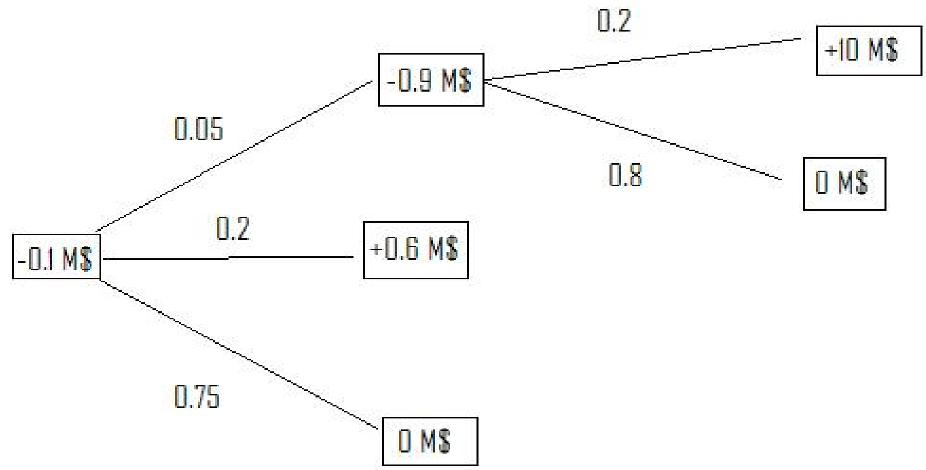
Problem 5 *Internet Startups* **[18 points]** Venture Capitalist Hit Wannabee believes it takes 0.1 million dollars to fund one internet startup for a year. For each startup that he funds, Mr. Wannabee estimates that there is a 75% chance that after a year it is bankrupt and worth nothing. He expects there is a 20% chance that the startup matures after one year into a mediocre performer which will be sold at that time for 0.6 million dollars. His estimate for the remaining 5% of startups is that he will have to invest 0.9 million dollars for the second year. And for these companies, 80% will be bankrupt after the second year, while 20% will become stars of Internet 2.0 and be acquired by either Microsoft or Google for 10 million dollars. Hit Wannabee believes change is the spice of life, and never holds a company more than two years. *For the purposes of this problem, assume Hit Wannabee's fantasies as above describe a hypothetical reality. Also, ignore interests rates for the period, i.e. profits he could have made by saving or investing somewhere else.*

a) : Create a probability model (probability table) for the net amount of money (i.e. profit - here negative means a loss) Hit Wannabee ends up with after creating one startup and letting it run to conclusion as above.

b) : Find the expected amount he'll end up with from one startup following your model in part a). And the standard deviation of the random variable describing this profit.

c) : Suppose Hit Wannabee finds the standard deviation for one startup too large; i.e. the investment too risky (too volatile). He decides to spread his risk by investing instead in two startups, each following the above model; their results are independent. For each startup, he provides only half of the funding (other investors take part) and then gets only half the profit. Find the mean and standard deviation of the random variable describing his total profit from this investment. Can he cut his risk, i.e. reduce his standard deviation while keeping the same expected profit ?

Soln: a) The appended probability tree (see figure) gives for each node, or branching point, the amount of money invested or gained in million dollars (M$), where investment is a negative amount and profit is a positive amount; these are written in small squares.



For each branch of the tree, the probability is written on the side in the usual manner. For each endpoint of the tree, the final profit is found by adding the sums invested or gained as we go along the particular path. The tree has four endpoints, each with a final profit, positive or negative (call these $x_1, \dots, x_4$, in units M$), and for each endpoint we have to find the probability of arriving there (i.e. $p_1, \dots, p_4$).

The lowermost endpoint (bankrupt after one year) is attained with probability $p_1 = 0.75$, and the total investment is lost, so $x = -0.1$. The second endpoint from below (sold after one year) is attained with probability $p_2 = 0.2$, and the profit is $x_2 = 0.6 - 0.1 = 0.5$. The third endpoint from below (new investment after one year, then bankrupt) is attained with probability $p_3 = 0.05 \cdot 0.8 = 0.04$, and the profit is $x_3 = -0.1 - 0.9 = -1$. The uppermost endpoint (new investment after one year, then sold to Microsoft or Google) is attained with probability $p_4 = 0.05 \cdot 0.2 = 0.01$, and the profit is $x_4 = -0.1 - 0.9 + 10 = 9$.

So if X is the random variable expressing the final profit (positive or negative) in M$, then X takes value x_i with probability p_i, for $i = 1, \dots, 4$, as given in the list:

$x_1 = -0.1$	$x_2 = 0.5$	$x_3 = -1$	$x_4 = 9$
$p_1 = 0.75$	$p_2 = 0.2$	$p_3 = 0.04$	$p_4 = 0.01$

b) We have

$$EX = \sum_{i=1}^{4} x_i p_i = (-0.1) \cdot 0.75 + 0.5 \cdot 0.2 + (-1) \cdot 0.04 + 9 \cdot 0.01$$
$$= 0.075.$$

Furthermore

$$\mathrm{Var}(X) = EX^2 - (EX)^2$$

$$= \sum_{i=1}^{4} x_i^2 p_i - (EX)^2$$
$$= (-0.1)^2 \cdot 0.75 + (0.5)^2 \cdot 0.2 + (-1)^2 \cdot 0.04 + 9^2 \cdot 0.01 - (0.075)^2$$
$$= 0.90188,$$

hence

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{0.90188} = 0.95.$$

c) Let X_1 and X_2 be the random profits from the two independent startups which follow the same probability model as X above and $T = \frac{1}{2}X_1 + \frac{1}{2}X_2$. Then

$$ET = \frac{1}{2}EX_1 + \frac{1}{2}EX_2 = EX = 0.075$$

$$\begin{aligned}\mathrm{SD}(T) &= \sqrt{\mathrm{Var}\left(\frac{1}{2}X_1\right) + \mathrm{Var}\left(\frac{1}{2}X_2\right)} = \sqrt{\frac{1}{4}2\mathrm{Var}(X)} \\ &= \sqrt{\frac{1}{2}}\mathrm{SD}(X) = 0.707 \cdot 0.95 = 0.672.\end{aligned}$$

Indeed he kept the same expected profit as for X but reduced the standard deviation (thus the risk) by a factor 0.707.

Problem 6 *Polygraphs* [**15 points**] Following an earthquake, 100 people are arrested on suspicion of looting and given a polygraph test. In past experience, the polygraph is 90 % reliable when administered to a guilty person and 98 % reliable when administered to an innocent person. Suppose that of the 100 people in custody only 12 are looters.
Hint: A tree diagram will be helpful for part of this problem.

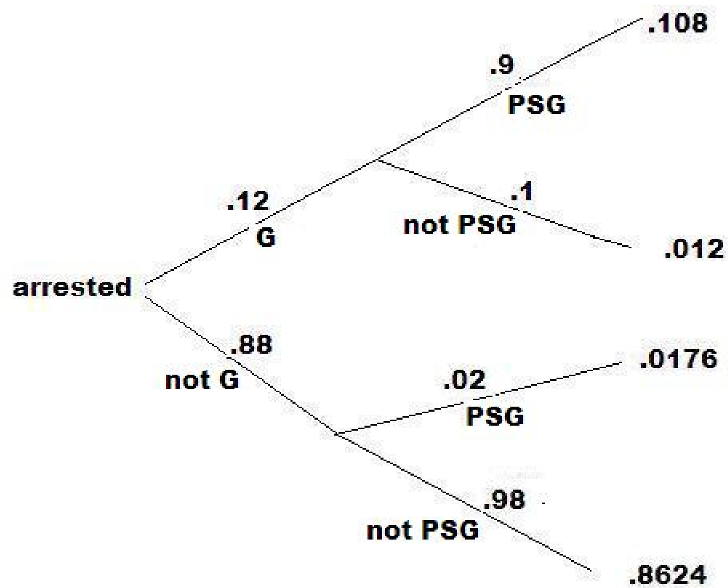
- **a:** What is the probability of a suspect being guilty of looting?
- **b:** Write the statement "the polygraph is 90 % reliable when administered to a guilty person" in terms of conditional probability. Please be sure and define any symbols or non-standard notation you introduce.
- **c:** Find the probability that a given suspect is innocent give that the polygraph says the suspect is guilty.

Soln:

Define the following events:

$$PSG = \text{polygraph says guilty;}$$
$$G = \text{ suspect is guilty.}$$

Below is a tree diagram for this situation:



(a). $P(G) = \dfrac{12}{100}$

(b). $P(PSG|G) = .90$

with PSG and G as defined at the beginning of the solution.

(c). $P(\text{not } G|PSG) = \dfrac{P(\text{not } G \cap PSG)}{P(PSG)} = \dfrac{.88(.02)}{.88(.02) + .12(.90)} = \dfrac{.0176}{.0176 + .108} = .14$