Math 304 **Spring 2007**

Glossary of Logical Terms[†]

The following glossary briefly describes some of the major technical logical terms used in this course. The glossary should be read through at the beginning and can then be consulted again as needed. The organization is logical rather than alphabetical.

1) In the beginning. . .

Definitions: In mathematics, we need to be as precise as possible with the terms that we use, since even one instance of imprecision can uncouple an entire chain of reasoning. So mathematicians make an effort to be very explicit and careful about the definitions of terms, particularly when working with something new, i.e., beyond the underlying context. A definition is an introduction and explanation of terminology or notation used in the subsequent mathematical text. Sometimes it can repeat a common, widely held understanding, similar to what we find in a dictionary definition. But, at other times, *it is an explicit device* used by the author/mathematician/student as a shorthand tool to refer to a particular new idea that they are working with. A definition is a matter of convention, to some extent arbitrary, expressing a definite understanding between author and reader which allows the mathematical conversation to proceed.

In presenting the terms that follow, we are giving definitions in the sense just described.

True and false statements: Mathematicians are constantly dealing with assertions (meaningful, declarative sentences) that correctly or incorrectly describe some underlying mathematical reality or context. Such an assertion is called a **true statement** when it does give a correct description and a **false statement** when it does not. In general, a meaningful, declarative sentence that is either true or false is called simply a **statement**. (Some mathematicians and logicians use the term "proposition" for this, but we will not.)

The two important aspects of this notion of a mathematical statement are: (1) *it is either true or false but not both*, and (2) *its truth (or its falsity) is a feature of the statement itself and does not depend on our ability to verify it.*

The word "true" (and the word "false") may also be used in a provisional sense, as when we assume that a certain statement is true and then reason based on this assumption. Paradoxically, such usage is valid even if our original assumption is counterfactual. For example, we may assume that a statement is true and then use reasoning to show that this assumption was incorrect. Indeed, this constitutes a useful technique of proof—proof by contradiction—that we'll be discussing further later.

Axiom: A foundational statement or basic stipulation or assumption, taken to be true and used as a basis for further reasoning.

Postulate: A stipulation or assumption similar to an axiom but with a slightly more provisional status.

2) Getting going...

Statements, axioms, etc., are static entities, but mathematics is dynamic. New mathematical statements are constantly being produced. It is, therefore, very important that there be clearly defined rules for deriving new statements from old.

Rule of inference: A rule that allows us to proceed from a given statement to a subsequent statement in such a way that whenever the given statement is true, then the subsequent statement is also true. (See **modus ponens** below.)

A caveat, however: if the given statement is false, then there is no predicting the truth or falsity of the subsequent statement.

A sequence of applications of rules of inference is sometimes called a **valid chain of reasoning**.

Proof: A proof is a method for establishing the truth of a mathematical statement by arriving at it via a valid chain of reasoning from one or more mathematical statements that are known to be true. One of the goals of this course is to learn various techniques for producing such valid chains of reasoning. A statement is called **provable** if it is the end result of a proof. Thus, every provable statement is true. Note, however, that we are not asserting that every true statement is provable.

3) Where are we headed?

The goal of mathematics is to obtain a certain kind of specialized knowledge, much of which is formulated in terms of true mathematical statements. Some of these statements can be very elementary or obvious and others can be highly complex or surprising. Certain terminology has been adopted to signify the relative importance of such statements.

Theorem: A theorem is a true, *significant* mathematical statement whose truth has been established by a proof. People may differ as to the significance of this or that true mathematical statement, but most mathematicians agree that, to qualify as a theorem, such a statement must have import or application beyond the immediate context in which it appears.

Proposition: This term is often used for a "lesser" theorem, that is, for a true mathematical statement of middle-level significance. The term is also sometimes used instead of the word "statement," as already mentioned, but we won't use it in this way in this course.

Lemma: This term refers to a true mathematical statement, established via a proof, whose main importance is that it forms a steppingstone to a proposition or a theorem. A lemma does not usually have significance beyond this and is often of an elementary character.

Corollary: This is usually a theorem that is an immediate consequence of another theorem. So, we say "Statement A is a corollary of Theorem B."

Conjecture: This term frequently appears in mathematical literature to designate a statement whose truth appears to the author to be very likely but which has not yet been established (by a proof).

Exercise 1. Which of the following are statements and which are not? Explain your choices.

(1) All men are mortal.
(2) $1 + 1 = 2$.
(3) $1 + 1 = 3$.
(4) Sam ate a sandwich.
(5) What time is it?
(6) Hands up!
(7) This sentence is false.
(8) The dog barked dream clouds of waxen votes.

Exercise 2. (A discussion problem.) Mathematics also proceeds by paradigms that do not explicitly involve axioms, inferences, proofs, and theorems. For example, much of mathematics is devoted to solving various kinds of equations. Consider, for example, the simple equation $2x + 3 = 7$. Can you show how solving this equation can be formulated as a theorem that can be proved?

4) Some of the pieces

Statements, whether they occur in mathematics or not, can often be broken into simpler **subsidiary statements** that are linked by certain **logical connectives**. The original statement is then sometimes called a **compound statement**. The following list describes these briefly; more details will be presented later.

Throughout the following, we let P and Q stand for any given statements.

Negation: *It is false that* P or, more briefly, *not*P. This asserts the contrary of what P asserts and is called the *negation* of (the subsidiary statement) P. (Negation is called a logical connective even though it applies to only one statement.)

For good English usage, we may have to modify the resulting statement. For example, if P is the statement "I can't hear you," then "It is false that I can't hear you" is grammatically correct but awkward, and "*not* I can't hear you" is not grammatically correct. So we may modify these to something like "I can hear you." Similar common-sense modifications may be used for the other connectives.

Conjunction: P *and* Q. This asserts *both* P *and* Q and is called the *conjunction* of (the subsidiary statements) P and Q.

For example, the conjunction of the two statements, "The moon was full last night" and "The Knicks beat the Bullets," is the assertion, "The moon was full last night and the Knicks beat the Bullets."

More generally, when a statement asserts *every one* of a set of subsidiary statements, we say that the statement is the *conjunction* of the subsidiary statements.

Disjunction: P *or* Q. This asserts that at least one of the two (subsidiary) statements P or Q is true and is called the *disjunction* of the statement P and the statement Q.

Notice that this notion of disjunction includes the case in which both P and Q are true and is sometimes called *inclusive disjunction.* In another form of disjunction, called *exclusive disjunction*, either one of P and Q is asserted but not both. For the most part, we do not use this form of disjunction, but when we do, we'll explicitly call it exclusive disjunction.

As an example of disjunction, we use the statements just discussed above: the statement "The moon was full last night *or* The Knicks beat the Bullets," is the disjunction of the statement "The moon was full last night" and the statement "the Knicks beat the Bullets."

More generally, when a statement asserts that *at least one* of a set of subsidiary statements is true, then we say that the statement is the *disjunction* of the subsidiary statements.

Implication: P *implies* Q. This asserts that Q is true whenever P is true. However, it asserts nothing about the truth or falsity of Q in the case that P is false.

Implication can have a variety of different English forms. In addition to "P implies Q," it is often expressed as "if P then Q." We'll mention other forms later.

Using the (admittedly silly) example given earlier, we can consider the implication "If the moon was full last night, then the Knicks beat the Bullets."

For the next five items, we shall be referring to the implication P *implies* Q.

Hypothesis: P is called the *hypothesis* of the implication.

Conclusion: Q is called the *conclusion* of the implication.

Converse: Q *implies* P is called the *converse* of the implication.

Inverse: *not*P *implies not*Q is called the *inverse* of the implication.

Contrapositive: *not*Q *implies not*P is called the *contrapositive* of the implication.

Tautology: Compound statements may be formed by repeatedly applying one or more of the foregoing connectives to various subsidiary statements. Sometimes, the very logical structure of the compound statement forces it to be true no matter what the truth status of the subsidiary statements. In this case, we call it a *tautology*. For example, the disjunction of the two statements "The moon was full last night" and "It is false that the moon was full last night" is a true statement whether or not the moon was full last night.

Contradiction: This is defined to be a compound statement which is the opposite of a tautology: it is always *false* no matter what the truth status of the subsidiary statements. Another way of saying this is that a contradiction is the negation of a tautology, and a tautology is a negation of a contradiction.

The simultaneous assertion of any statement and its negation, i.e, for any statement P, the statement P *and not*P, is an example of a contradiction, as follows immediately from our concept of truth. P *and not*P is perhaps the form of contradiction most widely used. Indeed, the literal meaning of the verb "to contradict" is to assert the contrary of (what has been asserted), which is precisely what P *and not*P does.

5) More terminology from the propositional calculus...

Truth-value: A numerical value, usually 1 or 0 assigned to a statement to indicate whether it is true or false. The truth-value of a compound statement can be algebraically expressed in terms of the truth-values of its subsidiary statements provided one uses 'mod two' arithmetic.

Truth-table: Because the truth-value of a compound statement S depends only on the truth-values of its subsidiary statements $S_1, S_2, \ldots, S_r$, we can construct a table that lists all 2^r possible truth-values for $S_1, S_2, \ldots$ and, next to each, lists the truth-value of S. This is called the truth-table of S.

Logical construction: A procedure that starts with some given statements and applies a finite number of logical connectives to them to obtain a compound statement.

Logical expression: A logical operation may be displayed symbolically by using logical connectives and parentheses, subject to various formation rules. This display is a logical expression.

Atomic statement, atom: A statement represented as an ingredient in a logical expression that is not viewed as being reducible into further subsidiary statements.

Logical equivalence: Two logical expressions are said to be logically equivalent if they have the same atoms and if any assignment of truth-values to these atoms produces the same truth-value for each expression. This can be checked by comparing truth-tables.

***Modus ponens*:** The most basic rule of inference: If statements A and $A \Rightarrow B$ are both true, then we may infer that B is true.