MATH 304: CONSTRUCTING THE REAL NUMBERS

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4 The Real Numbers

In the last section, we saw that the rational number system is incomplete inasmuch as it has gaps or "holes" where some limits of sequences should be. In this section we show how these holes can be filled and show, as well, that no further gaps of this kind can occur. This leads immediately to the standard picture of the reals with which a real analysis course begins. Time permitting, we shall also discuss material in the fifth and final section which shows that the rational numbers form only a tiny fragment of the set of all real numbers, of which the greatest part by far consists of the mysterious transcendentals.

For our basic construction now, we need to look carefully at different kinds of sequences of rational numbers, so that is where we start. We assume that students have some familiarity with these concepts from calculus courses.

4.1 Sequences of rational numbers

4.1.1 Definitions

Definition 1. a) As defined in the *Set Theory* notes, a sequence of rational numbers is function $f : \mathbb{N} \to \mathbb{Q}$. The function values $f(n)$ are called the *terms* of the sequence and are usually denoted by symbols such as a_n, b_n, or the like. That is, we write, say,

$f(n) = a_n$. The sequence f is then usually written as $\{a_n\}$. Sometimes, we may write out the sequence more fully as $\{a_0, a_1, a_2, \ldots\}$.

We let $\mathcal{S}$ denote the set of *all* sequences of rational numbers.

b) If U is some subset of $\mathbb{Q}$, we may say that $\{a_n\}$ is *in* U if every a_n is a member of U. We may also say then that the sequence *belongs to* U or *is contained in* U.

For example, suppose that U is some open interval of rational numbers with rational endpoints $c < d$. We borrow calculus notation to denote this by (c, d). More precisely, $U = (c, d)$ consists of all rational numbers s such that $c < s < d$. Then, we say that $\{a_n\}$ is in (c, d) if every a_n satisfies $c < a_n < b$ (equivalently $a_n \in (c, d)$).

(The interval notation (c, d) is the same as our ordered pair notation, but we use it anyway because it is the standard notation in calculus courses. We trust that the context will prevent confusion.)

> **Notational comment:** Occasionally, it will be more convenient to start the sequence with a_1, or a_2, or a_{n_0}, for some given natural number n_0, instead of with a_0. For example, if we are interested in the sequence of rationals $1, 1/2, 1/3, 1/4, \ldots$, it may be more convenient to write the general term as $a_n = 1/n$ and start with $n = 1$, rather than to write it as $a_n = 1/(n+1)$ and start with $n = 0$. This is mostly for notational convenience and will be the exception, not the rule. We assume that sequences begin at the "0 term" unless we explicitly state otherwise or unless such an initial term is clearly impossible. For an example of this last, if we write a sequence $\{1/n\}$ without further qualifications, this will mean that it starts at $n = 1$.

c) A sequence $\{a_n\}$ is called a *constant* sequence if all the terms a_n are equal. If their common value is, say, c, then we may say that $\{a_n\}$ is the *constantly c sequence.* For now, we denote the constantly c sequence by the symbol $\{c\}$. This is not to be confused with the set whose single member is the number c. The context will make it clear whether we are talking about the sequence or the set.

For the time being, we'll denote the set of all constant sequences by *Const.*

d) A sequence h in $\mathbb{N}$ is called *strictly increasing* provided $h(n) > h(m)$ whenever $n > m$. If we write $h(n) = h_n$, using sequence notation, then this means that $h_0 < h_1 < h_2 < \ldots$, etc.

If $f = \{a_n\}$ is a sequence of rationals, and h is a strictly increasing sequence in $\mathbb{N}$, then we say that the sequence given by the composition $f \circ h$ is a *subsequence* of f. Let us unravel this definition, using the sequence notation. For any $k \in \mathbb{N}$, we write the natural number $h(k)$ as h_k. Then, the composition $f \circ h(k)$ can be written as follows: $f \circ h(k) = a_{h(k)} = a_{h_k}$. The subsequence $f \circ h$ can then be written as $\{a_{h_k}\}$.

For example, if $\{a_n\}$ is the sequence $\{1/(n+3)\} = \{1/3,\ 1/4,\ 1/5,\ 1/6\ \ldots\}$, and the strictly increasing sequence of natural numbers h is given by $h(k) = 2^k$, then the corresponding subsequence is $\{1/(2^k + 3)\} = \{1/4,\ 1/5,\ 1/7,\ 1/11,\ \ldots\}$. For another example, suppose that $\{a_n\}$ is the following sequence of natural numbers: $0, 1, 0, 1, 0, \ldots$ and let h be the strictly increasing sequence given by $h_k = 2k$. Then, $a_{h_k} = 0$, for every k; i.e., this subsequence is just the constantly 0 sequence.

e) One useful application of the notion of subsequence is the concept of a *tail* of a sequence. Let $\{a_n\}$ be any sequence of rationals, and choose some natural number N. Define the strictly increasing sequence of natural numbers h_k by the rule $h_k = k + N$, for all natural numbers k. That is $\{h_k\}$ is the sequence $N, N+1, N+2, N+3, \ldots$. Then, the subsequence $\{a_{h_k}\}$ is the sequence of rational numbers $a_N, a_{N+1}, a_{N+2}, a_{N+3}, \ldots$, i.e., it is the original sequence with the first N terms deleted. We call this the N-*tail* of $\{a_n\}$, and we refer to any N-tail of $\{a_n\}$ simply as a *tail* of $\{a_n\}$. Notice that the 0-tail of $\{a_n\}$ is just $\{a_n\}$ itself, since deleting 0 terms doesn't affect anything.

We now use the notion of a tail of a sequence to define what we mean by convergent sequences and their limits.

Definition 2. Let $\{a_n\}$ be a sequence of rational numbers, and let L be a rational number. We say that the sequence $\{a_n\}$ *converges to* L if and only if the following holds: For every positive, rational number r, some N-tail of the sequence $\{a_n\}$ is contained in the open interval $(L-r, L+r)$. (In general, the particular N-tail so contained will depend on the choice of r. The smaller r is, the larger N will have to be.)

If $\{a_n\}$ converges to L, we call L the *limit* of the sequence, and we may express this by writing $\lim_{n\to\infty} a_n = L$ or $a_n \to L$.

A sequence that converges to some L is called *convergent*, or it is said to *have a limit*.

Let $\mathcal{K}$ denote the set of all convergent sequences.

The following points are important.

- If an N-tail of a sequence belongs to $(L-r, L+r)$, then (i) it also belongs to $(L-s, L+s)$, for every $s \geq r$, and (ii) any P-tail of the sequence also belongs to $(L-r, L+r)$, for $P \geq N$.

- Although the definition refers to any positive rational number r, the key feature of a convergent sequence is that *no matter how small r is*, some tail of the sequence is in $(L-r, L+r)$.

- The definition of convergence has another, equivalent, form that often appears in analysis or (some) calculus books. We express it here in symbolic-logic terms. The reader should convince himself/herself that this is, indeed, equivalent to the definition above:

  The sequence $\{a_n\}$ converges to L if and only if

$$(\forall r)(\exists N)(\forall n)\big(n \geq N \Rightarrow\ L - r < a_n < L + r\big).$$

We remind the reader that the pair of inequalities

$$L - r < a_n < L + r$$

is equivalent to the pair $-r < a_n - L < r$, which, in turn, is equivalent to

$$|a_n - L| < r.$$

Often, this last inequality is the one that is used because it is more compact. Using this, the reader should have no trouble with the following exercise.

Exercise 1. Let $\{a_n\}$ be a sequence of rational numbers, and let L be a rational number. For each natural number n, let $b_n = a_n - L$. Use Definition 2 to prove that $\lim_{n\to\infty} a_n = L$ if and only if $\lim_{n\infty} b_n = 0$. (We often write this last as $\lim_{n\to\infty}(a_n - L) = 0$.)

Exercise 2. Suppose that $\{a_n\}$is a sequence of rationals and that L and M are rational numbers such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$. Prove that $L = M$.

(Hint: Use proof by contradiction. Look at what happens to tails of the sequence if $L \neq M$.)

This exercise shows that if $\{a_n\}$ converges, its limit is unique. This is not asserted in the definition, so it must be proved. In the mathematical subject of General Topology, one studies convergence of sequences (defined similarly to the above definition) in mathematical systems (called topological spaces) in which limits need not always be unique. This will not play a role in what we do in this course.

4.1.2 Examples

We give a few simple examples of sequences, some of which converge and some of which do not. We also give some proofs to illustrate how to work with the definition of convergence.

a) The sequence $\{1/n\}$ converges to 0. Here is a proof. Suppose that r is any positive rational number. According to Theorem 4 of the *Rational Numbers* notes, there is a positive natural number N such that $1/N < r$. It is then easy to check that the N-tail of the sequence belongs to the interval $(-r, r)$.

b) The sequence $\{(-1)^n/n\}$ converges to 0. Virtually the same proof as in a) works for this sequence.

c) The sequence $\{(-1)^n\}$ does not converge. This assertion of non-convergence suggests that we try a proof by contradiction. To prepare for the proof, we notice that since the sequence oscillates between the values 1 and -1, the gaps between successive terms remains large (i.e., equal to 2). So, tails of the sequence will not fit into small intervals. With this preliminary observation, we can now do the proof by contradiction. Suppose a limit L did exist, and consider the interval $(L-1, L+1)$.

If a and b are in that interval, with, say, $a < b$, then $L - 1 < a < b < L + 1$, and so, $0 < b - a < 2$. But, in our sequence, $a_{2n} - a_{2n+1} = 2$, for every n, so it is impossible for both a_{2n} and a_{2n+1} to belong to the interval. Therefore no tail of the sequence is contained in the interval, contradicting our assumption that the sequence converges.

d) The sequence $\{n\}$ does not converge for the same reason as in c). Successive terms remain far apart (in this case they differ by 1). A similar proof (by contradiction) works in this case.

4.1.3 Cauchy sequences

In the examples above, notice that the proofs always make reference to a limiting value L—even when we are proving that the sequence has no limit! This is because a definite limit L is part of the definition of convergence. This is awkward for us, because we want to consider sequences that "should" converge but have no rational number as limit. We must find a condition for this that does not make reference to a limiting value.

The two non-convergent sequences above suggest how to do this. We should look for a condition that requires pairs of terms in the sequence to get close together. The correct idea was discovered by one of the leading French mathematicians of his era, Augustin Cauchy [pronounced kō′-shee](1789-1857), who is credited with developing much of what is now called the foundations of mathematical analysis.

Before we describe Cauchy's criterion, we introduce one helpful concept.

Definition 3. Let $\{a_n\}$ be a sequence of rationals, and let s be any positive rational number. We shall say that $\{a_n\}$ is *s-pinched* if *every* pair of terms of the sequence, say a_m and a_n, satisfy

$$|a_m - a_n| < s.$$

In other words, any two terms of the sequence are strictly less than s units apart.

Notice two simple features of the definition:

- If a sequence is s-pinched, then so is every subsequence.
- If a sequence is s-pinched, then it is also t-pinched, for every $t \geq s$. Therefore, in some sense, the smaller s is, the harder it is for a sequence to be s-pinched.

In the examples given earlier, whether or not the sequences converge, it is easy to see that they are all 3-pinched.

Exercise 3. Suppose that the sequence $\{a_n\}$ is r-pinched and that the sequence $\{b_n\}$ is s-pinched. Prove that each of the sequences $\{a_n + b_n\}$ and $\{a_n - b_n\}$ is $(r + s)$-pinched.

Definition 4. We call a sequence $\{a_n\}$ a *Cauchy sequence* provided that, *for every positive rational number* s, there is a tail of $\{a_n\}$ that is s-pinched. (Of course, the particular tail will depend on s.)

Let $\mathcal{C}$ denote the set of all Cauchy sequences of rational numbers.

Here are some points to pay attention to in this definition.

- The key point in the definition is that *no matter how small* the positive number s is, we can always find some tail of the sequence that is s-pinched.
- The definition, therefore, gives a precise and formal way of saying that the terms of the sequence get “closer and closer” together.

- However, the definition does not require that each term is closer to its successor than to its predecessor. We give an example of this below.
- Notice that if the N- tail of a sequence is s-pinched, then so is every P-tail, for $P \geq N$, since each such P-tail is a subsequence of the n-tail.

The reader should be able to see that, among the examples above, the convergent sequences are both Cauchy sequences, whereas the non-convergent sequences are not.

Here is an example of a Cauchy sequence in which some terms are farther from their immediate successors than from their immediate predecessors. Start with the sequences $\{1, 1/2, 1/3, 1/4, 1/5, 1/6, \ldots\}$ and $\{1, 1, 1/2, 1/2, 1/3, 1/3, \ldots\}$. The second is obtained from the first by simply repeating each term of the first twice. Now add the corresponding terms of the two sequences to obtain the sequence

$$\{1+1, \frac{1}{2}+1, \frac{1}{3}+\frac{1}{2}, \frac{1}{4}+\frac{1}{2}, \frac{1}{5}+\frac{1}{3}, \frac{1}{6}+\frac{1}{3}, \ldots\}.$$

Both sequences clearly converge to 0. Hence so does their sum. It then follows from the next proposition that this sum is a Cauchy sequence. However, notice that the difference between the third and fourth terms equals 1/12, whereas the difference between the fourth and fifth is larger than 1/6. Similar examples can be found arbitrarily far out in the sequence. So, the differences between terms don't squeeze together in a monotone way, but they do "eventually" squeeze together.

Exercise 4. Prove this last assertion.

Exercise 5. Prove the following:

a. If $\{a_n\}$ is a Cauchy sequence, then every N-tail of $\{a_n\}$ is a Cauchy sequence.

b. If some N tail of a sequence $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is a Cauchy sequence.

All the examples of Cauchy sequences given in this section so far, are convergent sequences. This is not a coincidence, as the following proposition shows.

Proposition 1. *If $\{a_n\}$ is convergent, then it is Cauchy.*

Proof. Suppose $\{a_n\}$ is convergent, and choose any positive rational number s. We must show that some tail of $\{a_n\}$ is s-pinched. Let L be the limit of the sequence. For any r, some tail of $\{a_n\}$ is in $(L - r, L + r)$. It is then easy to check that any two terms in the tail differ by $< 2r$, i.e., the tail is $2r$-pinched. So, we specialize r so that $2r = s$. Then, apply the preceding argument to this r, and conclude that the corresponding tail of $\{a_n\}$ is $2r$-pinched, i.e., s-pinched, as desired. $\square$

In light of this proposition, we can now describe the relationship between the different sets of sequences we have considered: *Const* (the constant sequences), $\mathcal{K}$ (the convergent sequences), $\mathcal{C}$ (the Cauchy sequences), $\mathcal{S}$ (all sequences): namely,

$$\mathit{Const} \subseteq \mathcal{K} \subseteq \mathcal{C} \subseteq \mathcal{S}.$$

All these inclusions except the second one follow immediately from the definitions; the second one is what is asserted by Proposition 1.

Exercise 6. Verify that the sequence in Theorem 5 of the *Rational Numbers* notes is a Cauchy sequence.

In general, the converse of this proposition is false. For example, we indicated in the *Rational Numbers* notes that the sequence in Theorem 5 does not converge, for if it did, the limit L would have to satisfy $L^2 = 2$, which no rational number can do. We did not give a rigorous proof of this equality then and will still have to defer that until a later subsection.

However, the following exercise describes cases in which one can prove that certain Cauchy sequences do converge.

Exercise 7. a) Let $\{a_n\}$ be a Cauchy sequence that has a subsequence which is constantly equal to the rational number c. Prove that $\{a_n\}$ converges to c. (Hint: Show that every tail of $\{a_n\}$ that is s-pinched must be contained in the interval $(c - s, c + s)$.)

b)* Let $\{a_n\}$ be a Cauchy sequence that has a subsequence converging to L. Prove that $\{a_n\}$ converges to L. (Hint: Show that every tail of $\{a_n\}$ that is s-pinched is in $(L - 2s, L + 2s)$.)

4.2 Algebraic operations on sequences

Definition 5. We define standard algebraic operations on $\mathcal{S}$ as follows: let $\{a_n\}$ and $\{b_n\}$ be any two sequences of rational numbers. Then

a. $\{a_n\} + \{b_n\} = \{a_n + b_n\}$;

b. $\{a_n\} - \{b_n\} = \{a_n - b_n\}$;

c. $\{a_n\} \cdot \{b_n\} = \{a_n \cdot b_n\}$;

d. If every $b_n \neq 0$, then $\{a_n\}/\{b_n\}$ is defined to be $\{a_n/b_n\}$.

4.2.1 The ring $\mathcal{S}$ and its subrings

The reader can easily verify the following proposition.

Proposition 2. *The set $\mathcal{S}$, together with the operations $+$ and $\cdot$ just defined, is a commutative ring. Its additive identity is the constantly 0 sequence, and its multiplicative identity is the constantly 1 sequence.*

Definition 6. Suppose that $< R, +, \cdot >$ is a commutative ring and that $\tilde{R}$ is a subset of R satisfying the following properties: (a) The additive identity 0 and the multiplicative identity 1 of R belong to $\tilde{R}$. (b) $\tilde{R}$ is *closed* under the operations of addition, subtraction, and multiplication. It is then easy to check that these operations, restricted to $\tilde{R}$, make $\tilde{R}$ into a commutative ring. It is called a *subring* of R.

Proposition 3. *Each of the sets Const, $\mathcal{K}$, and $\mathcal{C}$ is a subring of the commutative ring $\mathcal{S}$.*

In the case of *Const*, the proposition is almost obvious. The sum, product, and difference of constant sequences are again constant, and both 0 and 1 are constant. So *Const* is a subring.

The cases of $\mathcal{K}$ and $\mathcal{C}$ are less obvious. Of course, both the constantly 0 sequence and the constantly 1 sequence are convergent and are easily seen to be Cauchy, so condition (a) of the definition is satisfied. But, condition (b) is less clear, particularly in the case of the operation of multiplication. One has to show that the sum, difference and product of convergent sequences is convergent, and the sum, difference and product of Cauchy sequences is Cauchy. These facts for convergent sequences are proved in calculus courses, so we'll just state that result here and concentrate on Cauchy sequences.

Proposition 4. *Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences converging to limits L and M, respectively. Then $\{a_n\} + \{b_n\}$, $\{a_n\} - \{b_n\}$, and $\{a_n\} \cdot \{b_n\}$ are sequences converging to limits $L + M, L - M, L \cdot M$, respectively.*

It follows from this that $\mathcal{K}$ is a subring of $\mathcal{S}$.

Proposition 5. *Suppose that $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences. Then $\{a_n\}+\{b_n\}$, $\{a_n\} - \{b_n\}$, and $\{a_n\} \cdot \{b_n\}$ are also Cauchy sequences.*

It follows from this that $\mathcal{C}$ is a subring of $\mathcal{S}$. This implies that $\mathcal{C}$, together with the operations $+, -, \cdot$, is a commutative ring in its own right. Combining this with the previous proposition, we may conclude that $\mathcal{K}$ is a subring of $\mathcal{C}$.

4.2.2 Proving that $\mathcal{C}$ is a subring of $\mathcal{S}$

We first give an informal sketch of the proof of Proposition 5 in the case of addition. The key idea is the simple identity

$$(a_m + b_m) - (a_n + b_n) = (a_m - a_n) + (b_m - b_n).$$

We use this in the following way. Suppose that the N-tail of $\{a_n\}$ and the P-tail of $\{b_n\}$ are both s-pinched, for some positive rational number s. Choose the larger of the two natural numbers N and P, say P. Then the P tails of both $\{a_n\}$ and $\{b_n\}$ are both s-pinched, which means that $-s < a_m - a_n < s$ and $-s < b_m - b_n < s$, for $m, n \geq P$. Then, adding these inequalities and re-arranging terms according to the above identity, we get that $-2s < (a_m + b_m) - (a_n + b_n) < 2s$, for all $m, n \geq P$, which means that the P-tail of $\{a_n + b_n\} = \{a_n\} + \{b_n\}$ is $2s$-pinched. Now, let r be any positive rational. To find a tail of $\{a_n\} + \{b_n\}$ that is r-pinched, we just set $s = r/2$ and use the argument just concluded to get a P-tail of $\{a_n\} + \{b_n\}$ that is $2s = r$-pinched.

The proof for subtraction is almost the same, so we omit it. However, the proof for multiplication is harder. It requires an *additional idea*. Namely, we need to use the notion of a bounded sequence. A sequence $\{a_n\}$ is said to be *bounded* if there exists a positive rational number R such that $|a_n| < R$ for every n. The number R is then called a *bound* for the sequence. This is analogous to the notion of a bounded set: a set S of rational numbers is said to be bounded if there is a rational number, say A such that every $s \in S$ satisfies $|s| < A$.

The following fact will be needed. It looks obvious, but it requires an induction proof.

Exercise 8. Every finite set S of rational numbers is bounded. (Hint: Proceed by induction on the cardinality of S.)

Using this lemma, we can now prove the following:

Proposition 6. *Every Cauchy sequence is bounded*

Proof. Let $\{a_n\}$ be any Cauchy sequence. The sequence has a N-tail that is 1-pinched. It follows that, for $n \geq N$, every a_n satisfies $-1 < a_n - a_N < 1$, which is equivalent to $a_N - 1 < a_n < a_N + 1$. Since $a_N + 1 \leq |a_N| + 1$ and $a_N - 1 \geq -(|a_N| + 1)$, we get

$$-(|a_N| + 1) < a_n < a_N + 1,$$

provided $n \geq N$. This pair of inequalities is equivalent to $|a_n| < |a_N| + 1$. For $n < N$, we have the obvious inequalities $|a_n| < |a_n| + 1$.

Now we consider the finite set

$$S = \{|a_0| + 1, |a_1| + 1, |a_2| + 1, \ldots, |a_N| + 1\}.$$

Let R be any bound for this set, which exists by Exercise 8. The preceding inequalities imply that $|a_n| < R$ for all n. Thus, R is a bound for the sequence. $\square$

We have not yet explained why we need to consider bounds for sequences. Perhaps the best explanation involves a simple identity and an inequality. Recall that we want to prove that the product of two Cauchy sequences is a Cauchy sequence. That is, given two Cauchy sequences $\{a_n\}$ and $\{b_n\}$, we are interested in "pinching" absolute values of differences of the form $|a_m b_m - a_n b_n|$. If we add and subtract the term $a_m b_n$ inside the absolute-value, we obtain

$$|a_m b_m - a_m b_n + a_m b_n - a_n b_n| = |a_m(b_n - b_m) + (a_m - a_n)b_n|.$$

We now apply the "triangle inequality" to this last expression, obtaining

$$\begin{aligned} |a_m b_m - a_n b_n| &= |a_m(b_n - b_m) + (a_m - a_n)b_n| \\ &\leq |a_m| \cdot |(b_n - b_m)| + |(a_m - a_n)| \cdot |b_n|. \end{aligned}$$

Notice that the last expression on the right involves not only absolute values of differences, such as $|(b_n - b_m)|$, but also terms $|a_m|$ and $|b_n|$. These last terms can be very large. If they were unbounded, we might not be able to control the size of the terms $|a_m b_m - a_n b_n|$.

We can now sketch the proof that Cauchy sequences are closed under multiplication. Suppose that $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, and let us say that K is a bound for both $\{a_n\}$ and $\{b_n\}$. (We can always find such a common bound by just choosing bounds I and J for $\{a_n\}$ and $\{b_n\}$, respectively, and then letting K be the larger of these two.) Choose any positive rational number r. Then, there are tails of both $\{a_n\}$ and $\{b_n\}$ that are $r/2K$-pinched. We may assume that both tails are, say, P-tails by going further out in one of the sequences if necessary. Then, the above inequality implies that for m and $n \geq P$, we have

$$|a_m b_m - a_n b_n| < K \cdot (r/2K) + K \cdot (r/2K) = r,$$

as desired. That is, we have shown that the P-tail of $\{a_n\} \cdot \{b_n\}$ is r-pinched. So, the product of the two Cauchy sequences is Cauchy.

4.2.3 Comparing the rings with $\mathbb{Q}$

For any rational number r, let $j(r)$ denote the constant sequence $\{r\}$. Thus, j is a function $\mathbb{Q} \rightarrow \mathit{Const}$. It is obviously bijective and preserves all the operations and identities. So, we may use it to identify $\mathbb{Q}$ with Const. From now on, we dispense with the expression Const and use $\mathbb{Q}$ instead, and we denote the constantly r sequence simply by r. Occasionally, we shall want to draw a distinction between the number r and the constantly r sequence, but we shall do so explicitly then.

It follows that $\mathbb{Q}$ is a subring of the rings $\mathcal{K}$, $\mathcal{C}$, and $\mathcal{S}$.

4.2.4 Taking stock

Let us now summarize where we are. We have seen that $\mathbb{Q}$ is incomplete, so we seek to extend it to some kind of "nice" algebraic system in which holes are filled. As a first step, we constructed a commutative ring $\mathcal{S}$, the ring of all sequences of rational numbers. If the holes correspond to sequences of rationals that ought to converge, then certainly $\mathcal{S}$ contains these. However, $\mathcal{S}$ also contains sequences that cannot possibly be thought of as holes in $\mathbb{Q}$: for example, there are sequences that "go off to infinity," as well as sequences that oscillate in an undamped way.

As a next step, therefore, we restricted attention to sequences that converge and to sequences in which terms pinch together: i.e., we considered the rings $\mathcal{K}$ and $\mathcal{C}$.

As we have just seen, all of these are rings extending the field $\mathbb{Q}$.

There are still, however, two serious problems that remain.

- All of the rings except $\mathbb{Q}$ have zero divisors. Recall that this means that each of the rings contains pairs of non-zero elements whose product is zero. Here's a very simple example: Let $\{a_n\}$ be the sequence $\{1, 0, 0, 0, \ldots\}$ and let $\{b_n\}$ be the sequence $\{0, 1, 0, 0, \ldots\}$. Clearly neither equals the constantly zero sequence, but their product does. The reader can easily come up with many more such examples.

  This is bad, because we want our new number system to be a field, and a field cannot have zero divisors.

- All of the rings except $\mathbb{Q}$ are much too big. For example, there are infinitely many sequences in $\mathcal{K}$ that converge to any given rational number, say converge to 17. This is overkill by a long shot. Also, in $\mathcal{C}$, although we may not be able to talk about the limit of a given sequence—i.e., when it corresponds to some "hole" or hypothetical limit, we can nevertheless say when any two Cauchy sequences get "closer and closer" to each other: namely, their difference converges to 0. However, given any sequence $\{a_n\}$ in $\mathcal{C}$, it is easy to see that there are infinitely many other sequences in $\mathcal{C}$ which differ from $\{a_n\}$ by a sequence that converges to 0. All of these sequences should be viewed as determining the same hypothetical limit as $\{a_n\}$ does.

It turns out that we can solve both problems together by the same method, a method similar to the one we used earlier in constructing the integers and the rationals: namely, we use an equivalence relation to partition sequences into equivalence classes, with each class consisting of all sequences that "ought" to be considered the same. We now make this precise.

4.3 The field of real numbers

The key idea for solving the problems just outlined is to define a subset $\mathcal{Z}$ of $\mathcal{C}$ and use it to define an equivalence relation on $\mathcal{C}$.

$\mathcal{Z}$ is defined to consist of all sequences in $\mathcal{C}$ that converge to zero.

Proposition 4 immediately implies the following:

Proposition 7. *$\mathcal{Z}$ is closed under addition, subtraction, and multiplication. It contains the constantly* 0 *sequence.*

Exercise 9. Sketch a short proof of this proposition.

Since $\mathcal{Z}$ obviously does not contain the constantly 1 sequence, it is not a subring of $\mathcal{C}$. However, it has an additional property that is extremely useful.

Proposition 8. *Let $\{a_n\}$ be any sequence in $\mathcal{Z}$, and suppose that $\{b_n\}$ is any bounded sequence. Then, the product $\{a_n\} \cdot \{b_n\}$ belongs to $\mathcal{Z}$.*

Proof. Suppose that R is a bound for $\{b_n\}$ and let r be any positive rational number. Since $\{a_n\}$ converges to 0, some N-tail of $\{a_n\}$ belongs to $(-r/R, r/R)$. Therefore, for $n \geq N$, $|a_n b_n| = |a_n||b_n| < R|a_n| < R(r/R) = r$, which tells us that the N-tail of $\{a_n\} \cdot \{b_n\}$ belongs to $(-r, r)$. Since r is an arbitrarily chosen positive rational number, this means that $\{a_n\} \cdot \{b_n\}$ converges to 0. $\square$

Since, according to Proposition 6, every Cauchy sequence is bounded, we may apply Proposition 8 to the case where $\{b_n\}$ is a Cauchy sequence, which we do from now on.

Definition 7. Given two Cauchy sequences $\{a_n\}$ and $\{b_n\}$, we say that $\{a_n\}$ is *congruent* to $\{b_n\}$ *mod*$(\mathcal{Z})$, written

$$\{a_n\} \equiv_{\mathcal{Z}} \{b_n\},$$

provided that $\{a_n\} - \{b_n\} \in \mathcal{Z}$.

Theorem 1. *a. $\equiv_{\mathcal{Z}}$ is an equivalence relation on $\mathcal{C}$.*

b. Suppose that $\{a_n\}\equiv_{\mathcal{Z}}\{a'_n\}$ and $\{b_n\}\equiv_{\mathcal{Z}}\{b'_n\}$. Then,

$$\{a_n\} + \{b_n\} \equiv_{\mathcal{Z}} \{a'_n\} + \{b'_n\} \, and \, \{a_n\} \cdot \{b_n\} \equiv_{\mathcal{Z}} \{a'_n\} \cdot \{b'_n\}.$$

Comments:

- The notation a'_n here has nothing to do with the successor function. The symbol " $'$ " is simply used to distinguish a from a', etc.

- It is worth noting that the $\equiv_{\mathcal{Z}}$-equivalence class of the constantly 0 sequence is precisely the set $\mathcal{Z}$. This follows immediately from definitions, as the reader should check.

- Let us take $\{b'_n\}$ to equal the constantly 0 sequence, and $\{a_n\}$=$\{a'_n\}$ in the above theorem. Then the condition $\{a_n\} \equiv_{\mathcal{Z}} \{a'_n\}$ is true because $\equiv_{\mathcal{Z}}$ is reflexive, and the condition $\{b_n\} \equiv_{\mathcal{Z}} \{b'_n\}$ is equivalent to the statement $\{b_n\} \in \mathcal{Z}$, by the preceding comment. Therefore, in this case, Theorem 1 b) becomes the following: If $\{b_n\} \in \mathcal{Z}$, then $\{a_n\} + \{b_n\} \equiv_{\mathcal{Z}} \{a_n\}$ and $\{a_n\} \cdot \{b_n\} \in \mathcal{Z}$.

Exercise 10. a. Prove a) of the above theorem, and prove b) for the case of addition.

b. * Prove b) for the case of multiplication. (Hint: First prove the result in case $\{a_n\}$=$\{a'_n\}$. Then prove the result in the case $\{b_n\}$=$\{b'_n\}$. Both of these require Proposition 8. Then combine these cases somehow to deduce the general case.)

We can now define the set of real numbers, $\mathbb{R}$:

Definition 8. The set of real numbers, $\mathbb{R}$, is defined to be the set of equivalence classes $\mathcal{C}/\equiv_{\mathcal{Z}}$. That is, each real number is an equivalence class of Cauchy sequences, any two sequences in the same class differing by a sequence that converges to zero.

We can define operations of addition and multiplication on $\mathbb{R}$ by using the addition and multiplication already defined for sequences. Namely: Suppose r and s are real numbers, say $r = [\{a_n\}]$ and $s = [\{b_n\}]$, where we use the usual notation for equivalence classes. Then, we define

$$r + s = [\{a_n\} + \{b_n\}]$$
$$r \cdot s = [\{a_n\} \cdot \{b_n\}]$$

Statement (b) of the foregoing theorem shows that these operations are well-defined.

So much for definitions. What remains is the hard part. We need to check that $\mathbb{R}$, so defined, has all the properties we hope and expect the real numbers to have.

Proposition 9. *a. The operations $+$ and $\cdot$ defined on $\mathbb{R}$ make $\mathbb{R}$ into a commutative ring.*

b. Let $f : \mathbb{Q} \to R$ be defined by the rule $f(a) = [a]$, for every rational number a. Then: f is 1-1; f sends the additive and multiplicative identities of $\mathbb{Q}$ to those of $\mathbb{R}$; $f(a + b) = f(a) + f(b)$; $f(a \cdot b) = f(a) \cdot f(b)$.

Comments:

- Statement b) of the proposition shows that we may identify identify $\mathbb{Q}$ with a subring of $\mathbb{R}$.

  Exercise 11. Verify statement (b) of the proposition.

- The proof of statement (a) is not hard, given that the operations on $\mathbb{R}$ are well-defined. For each of the commutative-ring axioms, just use the fact that $\mathcal{C}$ satisfies that axiom. That will imply that the $\equiv_{\mathcal{Z}}$ classes satisfy it.

- So, we have started with a commutative ring, $\mathcal{C}$, that extends $\mathbb{Q}$, and ended up with another commutative ring, $\mathbb{R}$, that extends $\mathbb{Q}$ ($\mathbb{R}$ being the set of $\equiv_{\mathcal{Z}}$

classes of Cauchy sequences). What has this gained us? The answer is not yet apparent, but it will be shortly.

Exercise 12. Let $\{a_n\}$ be a Cauchy sequence, and let $\{b_n\}$ be a subsequence of $\{a_n\}$. Verify that $\{b_n\}$ is a Cauchy sequence and that $\{a_n\} \equiv_{\mathcal{Z}} \{b_n\}$.

This exercise tells us that if r is any real number, represented by a Cauchy sequence, say $\{a_n\}$ (i.e., r is the equivalence class of $\{a_n\}$), then any subsequence of $\{a_n\}$ also represents the same real number r. It is sometimes useful in proofs to use such a subsequence instead of the originally given Cauchy sequence.

So far, all we have shown is that $\mathbb{R}$ is a commutative ring extending $\mathbb{Q}$. It remains to verify that $\mathbb{R}$ is a field. The result that follows is our main tool for doing this. However, first, we introduce a piece of notation that is carried over from calculus courses. If a is any rational number, let (a, ∞) denote the set of all rational numbers $s > a$, and let $(-\infty, a)$ denote the set of all rational numbers $s < a$. (Note: we are *definitely not* suggesting that the symbols ∞ or $-\infty$ represent numbers.)

Lemma 1. *Suppose that $\{a_n\}$ is a Cauchy sequence that does not belong to $\mathcal{Z}$. Then, there is a positive rational number r and a natural number P such that the P-tail of $\{a_n\}$ either belongs to (r, ∞) or to $(-\infty, -r)$.*

Proof. Since $\{a_n\}$ does not converge to zero, there exists a positive rational number s such that *no* tail of $\{a_n\}$ belongs to $(-s, s)$. Since $\{a_n\}$ is a Cauchy sequence, there is an N-tail of the sequence that is $s/3$-pinched. For some $P \geq N$, a_P is not in $(-s, s)$. Say, $a_P \geq s$. If $n \geq P$, then $|a_P - a_n| < s/3$, and so $a_n > a_P - s/3 \geq 2s/3$. Therefore, the P-tail of $\{a_n\}$ is contained in $(2s/3, \infty)$. (In the event that $a_P \leq -s$, the same argument shows that the P-tail of $\{a_n\}$ is contained in $(-\infty, -2s/3))$. Thus, the conclusion of the lemma holds for $r = 2s/3$. □

It is useful to have a name for the two possibilities described in the conclusion of the lemma. If a Cauchy sequence $\{a_n\}$ has a tail that belongs to (r, ∞), for some positive rational number r, we shall describe this by saying $\{a_n\}$ is *very positive.* On the other hand, if it has a tail belonging to $(-\infty, -r)$, for some positive rational number r, then we'll say $\{a_n\}$ is *very negative.* In either case, the terms in the tail may be said to be "bounded away from zero."

Thus, the lemma can be paraphrased by saying that a Cauchy sequence that does not converge to zero is either very positive or very negative.

Notice that there are plenty of Cauchy sequences that are not very positive but have every term positive. For example, the sequence $\{1/n\}$ has every term positive, but it is not very positive because it converges to zero. Similar examples exist in the negative case.

Corollary 2. *Suppose that $\{a_n\}$ is a very positive Cauchy sequence such that every a_n is non-zero. Then, the sequence $\{1/a_n\}$ is a very positive Cauchy sequence. A similar result holds for very negative Cauchy sequences.*

Proof. Since every a_n is non-zero, the sequence $\{1/a_n\}$ is defined. Since $\{a_n\}$ is very positive, there is a positive rational r and an N-tail of $\{a_n\}$ contained in (r, ∞), in particular, for every $n \geq N$, $a_n > r$. Now, we make use of the following calculation, valid for all $m, n \geq N$:

$$|1/a_n - 1/a_m| = |a_n - a_m|/|a_n a_m| < |a_n - a_m|/r^2.$$

Finally, to show that $\{1/a_n\}$ is a Cauchy sequence, we choose any positive rational number s, and we choose a natural number $P \geq N$ such that the P-tail of $\{a_n\}$ is $r^2 s$-pinched. From the inequality above, it follows immediately that the P-tail of $\{1/a_n\}$ is $r^2 s/r^2 = s$-pinched, proving that the sequence $\{1/a_n\}$ is Cauchy.

To prove that $\{1/a_n\}$ is very positive, we note that $\{a_n\}$ is bounded, because it is Cauchy. Let R be a bound. Then, using the same N as above, if $n \geq N$, we have $r < a_n < R+1$, and so $1/a_n$ is defined and is $> 1/(R+1)$, which shows that $\{1/a_n\}$ is very positive. □

Corollary 3. *Suppose that $\{a_n\}$ is a Cauchy sequence that is very positive . Then, there exists a Cauchy sequence $\{b_n\}$ that is very positive and satisfies $\{a_n\} \cdot \{b_n\} \equiv_{\mathcal{Z}} 1$. Similarly if $\{a_n\}$ is very negative.*

Proof. We first prove the result in the case that every a_n is non-zero. Then, applying the above corollary, the sequence $\{1/a_n\}$ is a very positive Cauchy sequence, and clearly $\{a_n\} \cdot \{1/a_n\} = 1$, implying that $\{a_n\} \cdot \{1/a_n\} \equiv_{\mathcal{Z}} 1$. Setting $\{b_n\} = \{1/a_n\}$, we see that the corollary is proved.

If some a_n's are equal to 0, then we proceed as follows. Choose an N-tail of $\{a_n\}$ with each term positive. Define $b_n = 0$, for $n < N$, and $b_n = 1/a_n$, for $n \geq N$. Apply the above corollary to the N-tail of $\{a_n\}$ and the N-tail of $\{b_n\}$. It follows that the N tail of $\{b_n\}$ is a Cauchy sequence that is very positive. Therefore, $\{b_n\}$ is a Cauchy sequence that is very positive, and the N-tail of the product $\{a_n\} \cdot \{b_n\}$ equals the constantly 1 sequence. Therefore, the difference $\{a_n\} \cdot \{b_n\} - 1$ is a sequence whose N-tail is constantly 0. Clearly this converges to 0, and so $\{a_n\} \cdot \{b_n\} \equiv_{\mathcal{Z}} 1$, as desired. □

Corollary 4. $\mathbb{R}$ *is a field.*

We have already seen that $\mathbb{R}$ is a commutative ring, so all that remains to show is that every non-zero real number has a multiplicative inverse.

Exercise 13. Use the above corollaries to verify that every non-zero real number has a multiplicative inverse.

Therefore, we have successfully concluded the first stage of our construction of the reals. We have seen that $\mathbb{R}$ is an extension of $\mathbb{Q}$ and that it is a field. What remains for us is to check whether every Cauchy sequence of *rational numbers* (each rational now regarded as belonging to $\mathbb{R}$) converges to a limit in $\mathbb{R}$. We shall show this, but, actually, we shall show something substantially stronger: namely, we shall show that every Cauchy sequence of *real numbers* converges to a real number. This is the so-called *completeness property* of the real numbers. To discuss all this, we first have to define an ordering for the reals.

4.4 Ordering the reals

4.4.1 The basic properties

Definition 9. Let $\mathcal{C}^+$ consist of the very positive Cauchy sequences, and let $\mathcal{C}^-$ consist of the very negative Cauchy sequences.

Clearly these are disjoint sets, together consisting of all the Cauchy sequences that do not converge to 0.

Exercise 14.

a. Show that $\mathcal{C}^+$ and $\mathcal{C}^-$ are closed under addition and that $\mathcal{Z}$ and $\mathcal{C}^+$ are also closed under multiplication.

b. Show that the product of two sequences in $\mathcal{C}^-$ belongs to $\mathcal{C}^+$.

c. Show that the product of two sequences, one in $\mathcal{C}^+$ and the other in $\mathcal{C}^-$, belongs to $\mathcal{C}^-$.

d. Show that the sum of a sequence in $\mathcal{C}^+$ and a sequence in $\mathcal{Z}$ belongs to $\mathcal{C}^+$. Conclude that a similar fact holds when for $\mathcal{C}^+$ is replaced by $\mathcal{C}^-$.

Exercise 15. Show that if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences that are congruent mod($\mathcal{Z}$) and $\{a_n\} \in \mathcal{C}^+$, then the same holds for $\{b_n\}$. Similarly if $\{a_n\} \in \mathcal{C}^-$.

Definition 10. Let r be a real number, and let $\{a_n\}$ be a Cauchy sequence representing r (i.e., r is the equivalence class of $\{a_n\}$). We call r *positive* if $\{a_n\} \in \mathcal{C}^+$, and we call r *negative* if $\{a_n\} \in \mathcal{C}^-$. (Exercise 15 shows that this definition is well-posed.)

Let $\mathbb{R}^+$ be the set of all positive reals, and let $\mathbb{R}^-$ be the set of all negative reals. Clearly, $\mathbb{R}$ is the disjoint union of $\mathbb{R}^+$, $\mathbb{R}^-$, and $\{0\}$.

By Exercise 14, we may conclude that $\mathbb{R}^+$ is closed under addition and multiplication.

We now intend to make use of the subset $\mathbb{R}^+$ of $\mathbb{R}$ to define an order relation on $\mathbb{R}$ in the same way that we used $\mathbb{Q}^+$ to define an order relation on $\mathbb{Q}$. Theoretically, then, $\mathbb{Q}$ would have two order relations defined on it: one defined earlier and now this new one. In order to make sure that these are the same, we state the following:

Proposition 10. $\mathbb{Q}^+ = \mathbb{Q} \cap \mathbb{R}^+$

Let us unravel what this asserts. Recall that we first identified the rationals $\mathbb{Q}$ with the set *Const* of constant sequences. We then identified this with the $\equiv_{\mathcal{Z}}$-equivalence classes of constant sequences. This is how we came to regard $\mathbb{Q}$ as a subring (or, more informatively, a subfield) of $\mathbb{R}$. In this process, the positive rationals were identified with a certain subset of this subfield. Separately, we defined the set $\mathbb{R}^+$. The proposition asserts that those elements of $\mathbb{R}^+$ that happen to belong to the subfield of rationals are precisely the same as the rationals that were earlier defined to be positive. The following exercise indicates how this proposition should be proved.

Exercise 16. a. Prove that if c is a positive rational number, then the constantly c sequence is a very positive Cauchy sequence. (You may assume that the Cauchy property has already been proved earlier.)

b. Suppose that c is a rational number and $\{a_n\}$ is the constantly c sequence. Prove that if $\{a_n\}$ is very positive, then c is a positive rational number.

Implication a) immediately yields that $\mathbb{Q}^+ \subseteq \mathbb{Q} \cap \mathbb{R}^+$, whereas b) yields $\mathbb{Q} \cap \mathbb{R}^+ \subseteq \mathbb{Q}^+$. Together, these prove the proposition.

Of course, an exactly analogous fact holds for negative numbers.

We now define the desired order relation on $\mathbb{R}$.

Definition 11. Given real numbers r and s, we say that r is less than s, written $r < s$ (or, equivalently s is greater than r, written $s > r$) provided that $s - r \in \mathbb{R}^+$.

Notice that $r > 0 \Leftrightarrow r \in \mathbb{R}^+$ and $r < 0 \Leftrightarrow r \in \mathbb{R}^-$. We call such r *positive* and *negative*, respectively.

Exercise 17. Show that $<$ is well-defined.

We use the usual conventions concerning the ordering $\leq$ (and $\geq$) associated with $<$ (and $>$): that is, we write $r \leq s$ whenever we want to indicate that either $r < s$ or $r = s$. This is the (non-strict) linear ordering associated with $<$, as discussed in the *Set Theory* notes.

Definition 12. We define the *absolute value* function in the usual way, just as we did for rational numbers earlier : For any real number r, the absolute value $|r|$ is defined to be equal to r whenever $r \geq 0$, and it is defined to be equal to $-r$ whenever $r < 0$.

Exercise 18. a. Prove that, for any real numbers r and s, $|r| < s \Leftrightarrow -s < r < s$.

b. Use the definition of the absolute value function to prove the *triangle inequality*, which asserts that, for any real numbers r and s, $|r+s| \leq |r| + |s|$. (Hint: First show that $r + s \leq |r| + |s|$, and then show that $r + s > -(|r| + |s|)$.)

Working with inequality of reals is not completely straightforward, because one has to work with Cauchy sequences and ultimately show that some Cauchy sequence or another is very positive (or very negative) . The following exercise gives an example of this, and it has some useful applications besides.

Exercise 19. Suppose that $\{a_n\}$ and $\{b_n\}$ are two Cauchy sequences of rational numbers with the following property: For some positive rational number R, some N-tail of $\{a_n\} - \{b_n\}$ is in the interval $(-R, R)$. Let r and s be the real numbers given by $r = [\{a_n\}]$ and $s = [\{b_n\}]$. Show that $|r - s| < 2R$.(Hint: Show that the sequence $\{2R - (a_n - b_n)\}$ is very positive and that the sequence $\{-2R - (a_n - b_n)\}$ is very negative.)

We now state all of the properties for this ordering that we stated for the ordering of the rational numbers (Theorem 3 in the *Rational Numbers* notes). These are items b)–j) in the following theorem, and they are proved just as were the earlier properties. Item a) below shows that the order relation just defined on the reals extends that defined earlier on the rationals: it is proved using Proposition 10.

Theorem 2. *a. For any rational numbers a and b, $a < b$ in the ordering of $\mathbb{Q}$ if and only if $a < b$ in the ordering of $\mathbb{R}$.*

b. For any $r, s \in \mathbb{R}$, exactly one of the following holds: $r < s$, $r = s$, or $r > s$.

c. For any $r, s \in \mathbb{R}$, $r < s \quad \Leftrightarrow \quad -s < -r$.

d. For any $r, s, t \in \mathbb{R}$, $r < s \quad \wedge \quad s < t \quad \Rightarrow \quad r < t$; (transitivity of $<$).

e. For any $r \in \mathbb{R}$, $\neg(r < r)$. (reflexivity of $<$)

f. For all $r, s, t \in \mathbb{R}$, $r < s \quad \Leftrightarrow \quad r + t < s + t$.

g. If $r > 0$ and $s < t$, then $rs < rt$.

h. If $r < 0$ and $s < t$, then $rs > rt$.

i. For any $r, s \in \mathbb{R}^+$ such that $r < s$, we have $s^{-1} < r^{-1}$. Similarly, when $r, s \in \mathbb{R}^-$.

j. There exist non-empty subsets of $\mathbb{R}$ that are bounded below but contain no smallest element.

4.4.2 Density properties of the reals

The standard density property of the rationals is clearly echoed by the reals: namely, given any reals $r < s$, we can find a real number t such that $r < t < s$. Just use the average as before: i.e., set $t = (r + s)/2$. One needs only to verify that the stated inequalities hold, and both of these are implied by the inequality $(s - r)/2 > 0$, which is easily checked using statements (f) and (g) of the theorem.

There is, however, another density property that is very important and not so easily checked: namely, $\mathbb{Q}$ *is dense in* $\mathbb{R}$. We state this more precisely in the following proposition.

Proposition 11. *Let r and s be any real numbers such that $r < s$. Then, there exists a* rational number *c such that $r < c < s$.*

We shall call this property of the reals: the *rational density* property of $\mathbb{R}$.

This proposition is a little bit harder to prove than its counterpart in the case of rational numbers. To do so, it will be helpful first to prove the following simpler, related result.

Proposition 12. *Let r be any real number. Then there is a natural number N such that $r < N$.*

Proof. Suppose that $r = [\{a_n\}]$. Since $\{a_n\}$ is a Cauchy sequence, it is bounded, which means that there is a positive rational number R such that $|a_n| < R$. By a result proved in the *Rational Numbers* notes, there is a natural number M such that $R < M$. Therefore, $a_n < R < M$, or $a_n < M$, for every n. Let $N = M + 1$. Then $N - a_n > 1$, for every n, i.e., the sequence $\{N - a_n\}$ is very positive. It follows immediately that $N - r > 0$, or $r < N$, as desired. □

We now use this proposition to prove the rational density property of $\mathbb{R}$.

Proof. We suppose first that $r \geq 0$. Choose some natural number $P > s$ and a natural number $Q > 1/(s - r)$. Then consider the finite sequence of PQ natural numbers $1/Q, 2/Q, 3/Q, \ldots, PQ/Q = P$. The smallest of these, $1/Q$, is $< s - r$, and the largest, P, is $> s$, by construction. Consider all natural numbers n, such that n/Q is

$\geq s$. By what we just saw, this is a non-empty set of natural numbers. Let k be the minimum element of this set, so that $k/Q \geq s$ but $(k-1)/Q < s$. If $(k-1)/Q \leq r$, then $1/Q = k/Q - (k-1)/Q \geq s - r > 1/Q$, a contradiction. Therefore, the rational number $(k-1)/Q$ satisfies $r < (k-1)/Q < s$, as desired.

In case $r < 0$ and $s \leq 0$, we apply the same proof to the numbers $-r$ and $-s$, which satisfy $0 \leq -s < -r$. We get a rational number S such that $-s < S < -r$. Therefore, $-S$ is a rational number satisfying $r < -S < s$, as desired.

Finally, the only remaining case is the one in which $r < 0$ and $s > 0$, in which case, we have $r < 0 < s$, so that 0 qualifies as the rational number we are seeking. □

Here is an important application of rational density.

Corollary 13. *Let r and s be any real numbers. Then*

$$r = s \Leftrightarrow |r - s| < 1/M, \textit{for all natural numbers} \, M.$$

Exercise 20. Use Proposition 11 to prove Corollary 13.

Finally, we give one more application of rational density.

Corollary 14. *Let r and s be any two positive real numbers. Then, there is a natural number N such that $s < Nr$.*

We only sketch a proof. Apply rational density to the two real numbers 0 and r/s to find a rational number R such that $0 < R < r/s$, which implies $s < (1/R)r$. Then use Theorem 4 of the *Rational Numbers* notes to obtain a natural number $N > 1/R$, and replace $1/R$ by N in the preceding inequality.

As we indicated in the *Rational Numbers* notes, there are ordered fields in which the property described in this corollary is not satisfied: the so-called *non-Archimedean* fields. The corollary asserts that $\mathbb{R}$ is an Archimedean field.

4.4.3 Convergent and Cauchy sequences of reals

In order to discuss the more subtle properties of the order relation on $\mathbb{R}$, we need to carry over to the realm of real numbers the notions of convergent and Cauchy sequences defined earlier for rational numbers. This is easily done. All that is needed is to take the earlier concepts and replace the word "rational" by the word " real." Nevertheless, we quickly run through the relevant definitions again.

Definition 13. a. A sequence of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$. We usually write the function values $f(n)$ using subscripts, such as $f(n) = r_n$ and denote the sequence by $\{r_n\}$. The concepts of a constant sequence, a subsequence, and of an N-tail of a sequence are defined exactly as before, as is the notion of a sequence being s-pinched, for some positive real s.

b. Let $\{r_n\}$ be a sequence of real numbers, and let L be a real number. We say that $\{r_n\}$ *converges* to L, or that L is a *limit* of the sequence $\{r_n\}$, provided that, for every positive real number s, some N-tail of $\{r_n\}$ belongs to the interval $(L-s, L+s)$. Here we are extending the notation for intervals introduced in the *Rational Numbers* notes to include all reals: $(L-s, L+s)$ consists of all *real* r satisfying $L-s < r < L+s$.

When this is the case, we may write

$$\lim_{n\to\infty}\{r_n\} = L \quad \text{or} \quad \lim_{n\to\infty} r_n = L \quad \text{or} \quad r_n \to L.$$

c. A sequence of real numbers $\{r_n\}$ is called a *Cauchy sequence* provided that, for each positive real number s, some N-tail of $\{r_n\}$ is $s-pinched.$

Exercise 21. a. Suppose that $\{r_n\}$ is a sequence of real numbers, and L is a real number, such that for *every positive, rational number* s some N-tail or $\{r_n\}$ belongs to $(L-s, L+s)$. Conclude that $\{r_n\}$ converges to L. (Hint: Use the rational density property of the reals.)

b. Suppose that $\{r_n\}$ is a sequence of real numbers such that, for *every positive rational number* s, some N-tail of $\{r_n\}$ is s-pinched. Conclude that $\{r_n\}$ is a Cauchy sequence. (Hint: Use the rational density property of the reals.)

c. Let $\{a_n\}$ be a sequence of rational numbers. Suppose that it converges to a rational number L, in the sense defined in the *Rational Numbers* notes. Use a) to verify that it also converges to L as a sequence of real numbers as defined above.

d. Let $\{a_n\}$ be a sequence of rational numbers, and suppose that it is a Cauchy sequence, in the sense defined in the *Rational Numbers* notes. Use b) to verify that it is a Cauchy sequence of real numbers as defined above.

This exercise shows that the new definitions of convergent and Cauchy sequences of real numbers coincide with the old definitions when all the terms in the sequences happen to be rational.

Exercise 22. This exercise deals with sequences of real numbers.

a. Verify that every subsequence of a convergent sequence is convergent and has the same limit.

b. Verify that every subsequence of a Cauchy sequence is a Cauchy sequence.

c. Verify that a convergent sequence of real numbers has a *unique* limit. (Hint: Corollary 13 will be useful for this.)

Proposition 15. *Every convergent sequence of real numbers is a Cauchy sequence.*

This is proved just as its counterpart in the *Rational Numbers* notes.

We now come to the crucial point of this construction: the converse to Proposition 15. Recall that the corresponding converse in the *Rational Numbers* notes is false.

Theorem 3. *Every Cauchy sequence of real numbers converges.*

Proof. Let $\{r_n\}$ be a Cauchy sequence. The proof will have two parts. First, we construct a sequence of rational numbers $\{a_n\}$ approximating the sequence $\{r_n\}$ and prove that $\{a_n\}$ is a Cauchy sequence. Therefore, it represents a real number L, i.e., $L = [\{a_n\}]$. Second, we prove that $\{r_n\}$ converges to L.

(1) *The sequence* $\{a_n\}$. Define $a_0 = 0$. For any positive natural number n, use the rational density property of $\mathbb{R}$ to choose a rational number a_n satisfying

$$r_n < a_n < r_n + \frac{1}{n}. \tag{1}$$

To prove that $\{a_n\}$ is a Cauchy sequence, we use the following computation, valid for any positive integers m and n:

$$
\begin{aligned}
|a_m - a_n| &= |a_m - r_m + r_m - r_n + r_n - a_n| \\
&\leq |a_m - r_m| + |r_m - r_n| + |r_n - a_n| \\
&< |r_n - r_m| + \frac{1}{m} + \frac{1}{n}.
\end{aligned} \tag{2}
$$

Here, the first inequality is a double application of the triangle inequality, and the second inequality follows from the inequalities (1) above.

Now let any positive rational number s be given. By assumption, $\{r_n\}$ has an N-tail that is $s/3$-squeezed. Choose any natural number P that is greater than both N and $3/s$. Then, the P-tail of $\{r_n\}$ is $s/3$-squeezed and, if m and n are $\geq P$, we have $|r_m - r_n| < s/3$, $1/m < s/3$, and $1/n < s/3$. Using the computation (2), we have $|a_m - a_n| < s/3 + s/3 + s/3 = s$. That is, the P-tail of $\{a_n\}$ is s-squeezed. So, $\{a_n\}$ is a Cauchy sequence.

(2)$\{r_n\}$ *converges to* L. It will be important for this part of the proof, to be able to make a distinction between a rational number, say, a, and the constantly a sequence. We also need to avoid conflicting with our standard notation for sequences. So, we invent some new notation for constant sequences: we let, $c(a)$ stand for the constantly a sequence, and, for any natural number n, let $c(a)_n$ denote the n^{th} term of $c(a)$. Of course $c(a)_n = a$. Furthermore, as a constant sequence of rationals, $c(a)$ is a Cauchy sequence of rationals, so, when viewed this way, it represents a real number, which, of course, is again a. That is the equivalence class $[c(a)]$ equals a. So, we have a number of different ways of writing the rational number a: namely, as "a," as "$c(a)_n$," and as "[c(a)]."

Now, let any positive rational number s be given. We shall find a P-tail of $\{r_n\}$ that is contained in the interval $(L-s, L+s)$. According to Exercise 21, this will prove that $\{r_n\}$ converges to L.

Since, as we have shown in (1), $\{a_n\}$ is a Cauchy sequence, some N-tail of $\{a_n\}$ is $s/3$-squeezed: i.e., for $m, n \geq N$, we have $-s/3 < a_m - a_n < s/3$, which we rewrite as

$$-\frac{s}{3} < c(a_m)_n - a_n < \frac{s}{3}.$$

Now, by construction and definition, $[\{a_n\}] = L$ and $[c(a_m)] = a_m$. So Exercise 19 immediately gives the inequalities

$$-\frac{2s}{3} < a_m - L < \frac{2s}{3}, \tag{3}$$

valid for all $m \geq N$. Next we use the inequalities

$$-\frac{1}{m} < r_m - a_m < \frac{1}{m}, \tag{4}$$

valid for all positive m. If $m \geq s/3$, then $1/m < s/3$, so inequalities (4) imply

$$-\frac{s}{3} < r_m - a_m < \frac{s}{3}. \tag{5}$$

Finally, we choose $P \geq s/3$ and $\geq N$, and we choose any $m \geq P$ and add inequalities (3) and (5), obtaining,

$$-s < r_m - L < s.$$

Thus, the P-tail of $\{r_n\}$ is contained in the interval $(L-s, L+s)$, as desired. □

Not only have we shown that every Cauchy sequence of *rational* numbers converges in $\mathbb{R}$, so that *all the holes in the rationals are now plugged by reals*, but we have also shown that no new holes appear in the reals. The fact that all Cauchy sequences of reals converge to limits in the reals is known as *the completeness property of the real numbers.*

Since we have met all of our goals, we can call the construction of the reals complete.

4.4.4 A brief postscript

We have chosen to stop our construction at this point for reasons of time and space, but we should note that the story is not quite finished. As students well know, there are real polynomial equations without real roots. The example of $x^2 + 1 = 0$ is the simplest case of this. So, the reals are complete for the purposes of taking limits and doing calculus, but they are not algebraically complete. To take care of the case of the simplest example, i.e., the case of $\sqrt{-1}$, we can proceed as outlined for the case of $\sqrt{2}$: namely, we look at all expressions of the form $a + b\sqrt{-1}$ and define addition and multiplication in the obvious ways. Here, $\sqrt{-1}$ is regarded simply as a symbol. The result is a field extending $\mathbb{R}$, which we call the complex numbers and denote by $\mathbb{C}$.

Now when we did the analogous thing earlier with the rationals and $\sqrt{2}$, we obtained a field extending $\mathbb{Q}$. But we rejected this approach then partly because there were lots of other polynomial equations whose roots we also would need to adjoin. We also rejected it because we wanted all Cauchy sequences of rationals to converge to some real, and not all of these limiting values were necessarily roots of polynomials with rational coefficients. What about the situation now with $\mathbb{C}$?

We first note that there is no longer a problem with Cauchy sequences, essentially because our new, complex numbers are made up of two real pieces, and we know that Cauchy sequences of reals converge. (Of course, we omit details here.) The problem has to do with roots of polynomials, and here's where a magical thing happens. By just adjoining this one new root, $\sqrt{-1}$, to $\mathbb{R}$, we arrive at an extension field $\mathbb{C}$ that is algebraically complete. Technically, it is called *algebraically closed*. That is, *every degree-d polynomial with complex coefficients (in one variable) has d complex roots*! This assertion is the renowned *Fundamental Theorem of Algebra*, first proved by Carl Friedrich Gauss in 1797. This theorem has a large number of different proofs, each of which would require some weeks to develop fully.

4.4.5 Optional: Exercises on the foundations of calculus

Our construction of the reals leads immediately to some of the basic facts that are assumed or taken for granted in introductory calculus courses. This section is for the student who would like to see how some of these can be proved. The proofs are challenging exercises—with hints!— but worth a try for those planning to study analysis.

In many of the following exercises, there is an inductive construction of a sequence of closed intervals $I_0, I_1, I_2, \ldots$. The intervals are *nested*; i.e., $I_0 \supset I_1 \supset I_2 \supset \ldots$. Let us write $I_n = [a_n, b_n]$. Each interval after the first one will be chosen to be half of the predecessor interval. That is $I_n = [a_n, b_n]$ will be either $[a_{n-1}, (a_{n-1} + b_{n-1})/2]$ or $[(a_{n-1} + b_{n-1})/2, b_{n-1}]$. The particular choice will depend on the result we are trying to prove. We then will then let c_n equal some element of I_n. The choice will again depend on what we are trying to prove. It will not be hard to prove that

$\{c_n\}$ is a Cauchy sequence of reals. In one case, this concludes the argument. In some remaining cases, one has to show that the limit L of $\{c_n\}$ has a certain desired property.

Exercise 23. * Prove that every bounded sequence of reals contains a Cauchy subsequence.
(**Hint:** Let $\{r_n\}$ be the given sequence of reals. Since it is bounded, there is a positive real number R such that $\{r_n\}$ is contained in the closed interval $I_0 = [-R, R]$. Suppose we go through the construction of the nested sequence of subintervals as above, up to $I_{n-1} = [a_{n-1}, b_{n-1}]$ such that, for each of the constructed intervals, there are an infinite number of natural numbers k such that r_k belongs to the interval. In particular, this is true of I_{n-1}. Now divide I_{n-1} in half, as indicated above, and choose a half that contains an infinite number of terms of the sequence. At least one of the two halves must contain an infinite number of terms, by a prelim problem. If both halves do, then choose either half. This defines the sequence of intervals. Now, for any natural number k, choose r_{n_k} in I_k such that the subscript n_k is larger than that of any previously selected term. This can be done because I_k contains infinitely many terms of the sequence. Conclude the proof by showing that the selected subsequence is Cauchy.)

Exercise 24. * Prove that if S is a non-empty set of real numbers with an upper bound, then there exists a least upper bound (l.u.b.) of S. (A similar result asserts that if S has a lower bound, then it has a greatest lower bound (g.l.b.).)
(**Hint:** Start by choosing an number a_0 that is not an upper bound of S and an number b_0 that is an upper bound of S. The element a_0 can be chosen to be any number that is less than some element of S. Clearly $a_0 < b_0$. Set $I_0 = [a_0, b_0]$, and

construct the sequence of intervals I_n and let c_n be the midpoint of I_n. To choose I_n, use the following rule. Suppose that I_{n-1} has been defined, with a_{n-1} not an upper bound of S and b_{n-1} an upper bound of S. Of course, then $a_{n-1} < b_{n-1}$. If the midpoint $c_{n-1} = (a_{n-1}+b_{n-1})/2$ is *not* an upper bound of S, then let $I_n = [c_{n-1}, b_{n-1}]$. If it is an upper bound of S, let $I_n = [a_{n-1}, c_{n-1}]$. Now prove that $\{c_n\}$ is a Cauchy sequence, and show that its limit is the desired l.u.b.)

Exercise 25. *

This result concerns continuous functions. Recall that if f is a continuous real-valued function of a real variable, and if $\{a_n\}$ is a sequence converging to L, then $\lim_{n\to\infty} f(a_n) = f(L)$. (This fact is an easy consequence of the definition of continuity, and it is often used in calculus courses. We assume it here without proof. It is the only fact about continuous functions that will be needed for this exercise.)

Prove: Suppose that f is a continuous real-valued function with domain the closed interval $[a, b]$ of real numbers, and assume that $f(a) < f(b)$. Then for any real number r satisfying $f(a) < r < f(b)$, there is a real number c satisfying $a < c < b$ such that $f(c) = r$. This is known as the *Intermediate Value Theorem.*
(**Hint:** Start as before with $I_0 = [a, b]$ and construct the intervals I_n and mid-points c_n as before, subject to the following rule: If $f(c_{n-1}) \leq r$, let $I_n = [c_{n-1}, b_{n-1}]$. If $f(c_{n-1}) > r$, let $I_n = [a_{n-1}, c_{n-1}]$. Then show that $\{c_n\}$ is a Cauchy sequence and that its limit, which we call c, is contained in the open interval (a, b) and satisfies $f(c) = r$.)

We conclude with two more results.

Exercise 26. * Let f be a continuous real-valued function with domain a closed interval $[a, b]$ of real numbers. Then the range $R(f)$ is a bounded set of reals.

(**Hint:** Proceed by contradiction. If $R(f)$ is unbounded, find a sequence $\{x_n\}$ in $[a, b]$ such that $|f(x_n| > n$. Choose a convergent subsequence of $\{x_n\}$, and let c be a limit of this subsequence. Show that $c \in [a, b]$ and conclude the impossible assertion that $f(c) > n$, for every natural number n.)

Exercise 27. * Let f be a continuous real-valued function with domain a closed interval $[a, b]$ of real numbers. Then f achieves its maximum value on $[a, b]$. That is, there exists a $c \in [a, b]$ such that $f(c)$ is the maximum value of f.

(**Hint:** By the preceding exercise, $R(f)$ is bounded, and so it has a least upper bound y. Show that there must exist a convergent subsequence $\{x_n\} \in [a, b]$ such that $\lim_{n\to\infty} f(x_n) = y$.)