MATH 3320, SOLUTION #2

DUE FRIDAY, SEPTEMBER 7

To ensure that you get full credit, be sure to *show your work* in the problems that require calculations. Very little credit is given for answers without justification. Please write in complete sentences to help us understand what you are doing.

You may collaborate with classmates in solving the problems, including the extra credit problems. If you do so, please list their names on your assignment. However, you should not consult *any* other people (except the instructors or TAs), or use online resources. (Seriously, it's very obvious to us when this occurs, and there are drastic consequences, so don't do it!) If you use results that were not proved in class, please provide your own proof.

1. (15 points) Let n be a positive integer. Let $\sigma_0(n)$ denotes the number of positive divisors of n. (This is not the usual divisor function. For example, since 6 has positive divisors 1, 2, 3, and 6, we have $\sigma_0(6) = 4$.) Prove the following identity:

$$\prod_{\substack{d|n \\ d>0}} d = n^{\sigma_0(n)/2}.$$

In other words, the product of all positive divisors of n is equal to $n^{\sigma_0(n)/2}$.

Solution: Let $S = \{d : d|n, d > 0\}$. Consider $f : S \mapsto S$ where $f(d) = n/d$. This mapping is well defined because if $d|n$, n/d must be an integer, and $n/(n/d) = d \in \mathbb{N}$, which means $n/d \in S$. We claim f is a bijection, because

1. If $f(d_1) = f(d_2)$, then $n/d_1 = n/d_2 \Rightarrow d_1 = d_2$.
2. For all $d \in S$, we have $n/d \in S$, and $f(n/d) = d$.

Therefore,

$$\left(\prod_{d\in S} d\right)^2 = \left(\prod_{d\in S} d\right) \cdot \left(\prod_{d\in S} f(d)\right) = \prod_{d\in S}(d \cdot n/d) = \prod_{d\in S} n = n^{\sigma_0(n)}.$$

$$\prod_{d\in S} d = n^{\sigma_0(n)/2}.$$

Comment: You can also pair the factors of n instead of constructing the bijection and computing the square of the product. However, if you do so, be mindful about the notations and the case when n is a perfect square.

2. Let n be a positive integer. Let $\sigma_0(n)$ be as in the previous problem.

(a) (10 points) Prove that the number of positive integer solutions of $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$ is $\sigma_0(n^2)$.

Solution: This is a hard problem, but it is helpful to think carefully about how we can obtain some sort of correspondence between positive integers dividing $\sigma_0(n^2)$ and solutions to the equation.

Observe first that the given equation is equivalent to requiring $n(x + y) = xy$. Let $m = \gcd(x, y)$, and let $x_0 = x/m$, $y_0 = y/m$. Then picking a solution (x, y) may be equivalently viewed as picking a positive integer m and relatively prime positive integers x_0, y_0 such that (mx_0, my_0) is a solution. We may express this condition by substituting into our original equation. We obtain $mn(x_0 + y_0) = m^2x_0y_0$, which simplifies to $n(x_0 + y_0) = mx_0y_0$. In other words, it is equivalent simply to pick some relatively prime positive integers x_0 and y_0 for which x_0y_0 divides $n(x_0+y_0)$.

At this point I claim there is a bijective (i.e. one-to-one, onto) correspondence between such pairs (x_0, y_0) and positive integer divisors of n^2. Denote the first set by S and the second by T. Consider the maps

$$(x_0, y_0) \mapsto (n/y_0)x_0 \quad \text{and} \quad d \mapsto (d/\gcd(d,n), n/\gcd(d,n)).$$

First note that the left map successfully sends elements of S to elements of T and that the right map successfully sends elements of T to relatively prime positive integers, and that these integers $d/\gcd(d,n)$ and $n/\gcd(d,n)$ satisfy $(d/\gcd(d,n))(n/\gcd(d,n))|n(d/\gcd(d,n)+n/\gcd(d,n))$ (as this condition simplifies to $d/\gcd(d,n)|d+n$ which is true since $d|n^2 \Rightarrow \mathrm{lcm}(d,n)|n^2 \Rightarrow dn/\gcd(d,n)|n^2 \Rightarrow d/\gcd(d,n)|n$), and so sends elements of T to elements of S. It is further the case that these maps compose (in both directions) to yield the identity. In other words, if we start with an element of S and apply the left map followed by the right map, we end up with the element of S that we started with, and similarly with an element of T (followed by the right and then the left map). This may be fairly easily checked.

We conclude that the number of positive integer solutions to the original equation is equal to the number of positive divisors of n^2, i.e. to $\sigma_0(n^2)$.

(b) (10 points) If n is odd, prove that the number of integer solutions of $x^2 - y^2 = n$ is $2\sigma_0(n)$.

Solution: We factor, to obtain $(x+y)(x-y) = n$. Observe that, given any positive divisor d of n, if we set $x+y = d$ and $x-y = n/d$, we end up with exactly one solution, as x and y may be read off from these equations, and, due to the fact that d and n/d are odd, the values for x and y so determined are integers. On the other hand, we may alternatively set $x+y = -d$ and $x-y = -n/d$, and obtain two other solutions. Certainly these solutions are distinct, and also differ from one another over all choices of d, since the values for x and y add in each case to a different number. This process yields a total of $2\sigma_0(n)$ solutions.

On the other hand, suppose that (x, y) is any solution to $x^2 - y^2 = n$. Then $(x+y)|n$ and $x-y = n/(x+y)$, so this solution was already counted when we set d (or $-d$) equal to $x+y$. It follows that there are $2\sigma_0(n)$ total solutions.

(c) (5 points) Does (b) hold when n is even?

Solution: We claim that $n = 2$ is a counterexample. While $2\sigma_0(n) = 2\sigma_0(2) = 4$, we see that, if we set $x^2 - y^2 = 2$, we must have $2x = (x+y)+(x-y)$ equal to either $1+2 = 2+1 = 3$ or $-1-2 = -2-1 = -3$ (by factoring, and examining the possibilities). But this means that x would have to equal $\pm 3/2$, so there are no integer solutions. Since $0 \neq 4$, we have found a counterexample to the result from (b) holding for even n.

3.

(a) (10 points) Let a and b be relatively prime positive integers whose product ab is a perfect square. Show that a and b are both perfect squares.

Solution: Since ab is a perfect square, we may write it as c^2, where c is a non-negative integer. We use the fundamental theorem of arithmetic to write c in its prime power factorization:

$$c = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}.$$

Thus

$$ab = p_1^{2a_1} p_2^{2a_2} \cdots p_n^{2a_n}.$$

Since a and b are relatively prime, they do not share any prime factor. Hence, the prime power factorizations of a and b may be described by partitioning the set P of primes $p_1, \ldots, p_n$ into the set P_a of primes that divide a and the set P_b of primes that divide *b.* Now

$$a = \prod_{p_i \in P_a} p_i^{2a_i} \quad \text{and} \quad \prod_{p_j \in P_b} p_j^{2a_j}.$$

It follows that a and b are perfect squares.

(b) (10 points) Solve the Diophantine equation $x^2 = y^3 + y$.

Solution: Suppose we have a solution (x, y). Since $y(y^2+1) = x^2$ and since $\gcd(y, y^2+1) = \gcd(y, 1) = 1$, we know by part (a) that both y and y^2+1 are perfect squares. In particular, let us write $y^2+1 = u^2$ where u is an integer. Then $(u-y)(u+y) = u^2 - y^2 = 1$, so either $u - y = u + y = 1$ or $u - y = u + y = -1$. In either case, we obtain that $y = 0$. Then $x = 0$ as well, so the solution must be trivial. On the other hand, $(x, y) = (0, 0)$ is evidently a solution. We have thus solved the Diophantine equation.

4. (15 points) Prove that the equation $x^2 + y^2 + z^2 = 8xy$ has no nontrivial solutions. (Hint: Look at the equation modulo 4. What does that imply about x, y, and z? Show that this implies that x, y, and z must be all be infinitely divisible by some integer and so must be 0.)

Solution: Modulo 4 the equation reads $x^2 + y^2 + z^2 \equiv 0$. But observe that for any integer a, its square a^2 is congruent to either 0 or 1 mod 4. Since the left-hand side of our equation is just a sum of three squares, its value mod 4 is simply the number of integers x, y, z whose squares are congruent to 1. The only way this can be congruent to 0, then, is if $x^2 \equiv y^2 \equiv z^2 \equiv 0$ mod 4. But this implies that x, y, and z are all even. We may thus set them equal to $2x'$, $2y'$, and $2z'$, respectively, where x', y', and z' are integers. Now the fact that (x, y, z) satisfies the initial equation may be expressed as $(2x')^2 + (2y')^2 + (2z')^2 = 8(2x')(2y')$. Dividing by 4 on both sides, we can rewrite this as $x'^2 + y'^2 + z'^2 = 8x'y'$, thus showing that (x', y', z') must also be a solution to our equation. (Observe that we are able to recover the original equation through a method like this precisely because the equation was homogeneous—making it appropriate to factor out the same power of two at each term.) Since we may repeat this process arbitrarily many times, it follows that x, y, and z must each be divisible by arbitrarily large powers of 2. But every nonzero integer has at most finitely many factors of 2, so x, y, and z must all be zero. It follows that $x^2 + y^2 + z^2 = 8xy$ has only the trivial solution.

5. (15 points) Find all the primitive integer solutions of $x^2+y^2 = 2z^2$. (Hint: There are at least two possible approaches. One is to mimic the proof of the theorem on primitive Pythagorean triples. Another is to prove directly that $(\frac{x+y}{2}, \frac{x-y}{2}, z)$ is a primitive Pythagorean triple.)

Solution: For a primitive integer solution, x and y should both be odd. Otherwise, if one of them is odd and the other one is even, then $2 \nmid x^2 + y^2$ but $2|2z^2$. If both x and y are even, then $4|x^2 + y^2 = 2z^2 \Rightarrow 2|z^2$, and z is also even, contradicting with the assumption (x, y, z) is primitive.

In this case, $\frac{x \pm y}{2}$ are integers, and

$$\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 = \frac{x^2}{4} + \frac{xy}{2} + \frac{y^2}{4} + \frac{x^2}{4} - \frac{xy}{2} + \frac{y^2}{4} = \frac{1}{2}(x^2 + y^2) = z^2$$

The primitive integer solutions of $((x+y)/2, (x-y)/2, z)$ are

$$\begin{cases} \frac{x+y}{2} &= 2pq \\ \frac{x-y}{2} &= p^2 - q^2 \\ z &= p^2 + q^2 \end{cases},$$

$$\begin{cases} x &= p^2 + 2pq - q^2 \\ y &= -p^2 + 2pq + q^2 \\ z &= p^2 + q^2 \end{cases}$$

where one of p, q is even and the other is odd.

6. (Extra Credit) Below is a modified version of a test I made up when I was 18. It predicts the age that is best for you to get married. (Obviously, don't take this too seriously.)

Step 1. Select the number corresponding to your favorite month. (1 for January, 2 for February, ..., 12 for December)
Step 2. To this number, add your current age in years.
Step 3. Multiply this number by 3 and take the sum of the digits.
Step 4. Again, multiply the number you have by 3, and take the sum of the digits.
Step 5. Add the century that we're in now (If 1999 was the last year of the 20th century, we know that 2018 is in the...).
Step 6. From this number, subtract the number of children you would like to have.
Step 7. The resulting number is your ideal age for marriage.

For (10 points), discover, with proof, the secret behind this test.

Solution:

Lemma: an integer is congruent to its sum of digit mod 3 and mod 9.

Proof. Let the integer $n = a_1 a_2 \cdots a_k$ in digits. Then numerically

$$n = \sum_{i=1}^{k} a_i 10^{k-i}$$

$$n - \sum_{i=1}^{k} a_i = \sum_{i=1}^{k} a_i(10^{k-i} - 1)$$

Because $10^{k-i} - 1 = 9\sum_{j=0}^{k-i-1} 10^j$ is divisible by 9, we have $9|n - \sum_{i=1}^{k} a_i$. Because $3|9$, we also have $3|n - \sum_{i=1}^{k} a_i$. $\square$

Denote the resulting number of step k as n_k.

In this problem, considering the length of human life, $3(month + age)$ is at most a three digit number. n_3 is no larger than $9 \times 3 = 27$, and $3|n_3$.

Since n_4 is the sum of digit of $3n_3$, and $9|3n_3$, we know $9|n_4$. Because $n_3 \leqslant 27$, we have $3n_3 \leqslant 81$. For all positive integers no larger than 81 and divisible by 9, their sum of digits are all equal to 9. So $n_4 = 9$.

Then $n_5 = 21 + n_4 = 30$. Since giving birth to a child takes approximately 1 year, if we expect to have that many children before 30, subtracting 30 by the number of children will give a reasonable age for marriage.