MATH 3320, HOMEWORK #1

DUE FRIDAY, AUGUST 31

To ensure that you get full credit, be sure to *show your work* in the problems that require calculations. Very little credit is given for answers without justification. Please write in complete sentences to help us understand what you are doing.

You may collaborate with classmates in solving the problems, including the extra credit problems. If you do so, please list their names on your assignment. However, you should not consult *any* other people (except the instructors or TAs), or use online resources. (Seriously, it's very obvious to us when this occurs, and there are drastic consequences, so don't do it!) If you use results that were not proved in class, please provide your own proof.

1.

(a) (5 points) Use the Euclidean algorithm to find $\gcd(1287, 403)$.
(b) (10 points) Find all the integer solutions of $1287x + 403y = 104$.

2.

(a) (5 points) Give a definition for the greatest common divisor of three integers a, b, c.
(b) (15 points) Prove that $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$ for any integers a, b, c.
(c) (5 points) Use the Euclidean algorithm to find the greatest common divisor of 408, 884, and 1071.
(d) (10 points) Do there exist integers x, y, z such that $408x + 884y + 1071z = 123$? (Hint: You don't have to solve the equation.)

3. (10 points) Find all the integer solutions of $6x + 15y + 10z = 8$.

4. (15 points) Determine the number of *positive* integer solutions of $2x + 3y = 300$.

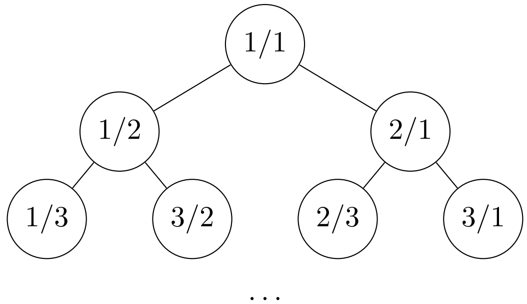
5. Recall that the Fibonacci sequence $\{F_n\}_{n\geq 1}$ is defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n$$

for $n \geq 1$, with initial values $F_1 = 1$ and $F_2 = 1$.

(a) (10 points) Find $\gcd(F_{n+2}, F_n)$.
(b) (10 points) Show that $\gcd(F_{n+3}, F_n) = \gcd(F_n, 2)$ for $n \geq 1$.
(c) (5 points) Use (b) to prove that F_{3m} is an even number for $m \geq 1$.

6. (Extra Credit) Consider a binary tree obtained by starting with the fraction $1 = \frac{1}{1}$ and iteratively adding $\frac{a}{a+b}$ and $\frac{a+b}{b}$ below each fraction $\frac{a}{b}$ as "children". For example, the top of such a tree looks like this:



and keeps on going. It is infinitely long and infinitely wide, and every node corresponds to a rational number.

Prove the following properties of this tree:

(a) (10 points) Every fraction in this tree is in reduced form (i.e. its denominator and numerator are relatively prime).

(b) (20 points) Every positive rational number appears exactly once in this tree.